Introduction to Information Sciences

Information Theory
Shannon's Entropy

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Summary of Today's Lecture

- Shannon's framework for information
- Shannon's entropy

Starting point...

> `Not everything that can be counted counts,`
> `and not everything that counts can be counted.`
> (Einstein)

- For things which ***can* be counted**, the science which provides a framework to
  - **store** *(efficiently)* that information,
  - **communicate** *(efficiently)* that information between individuals/computers,

  is a branch of

  - mathematics,
  - statistics,
  - electrical engineering, *etc.*

  called **information theory**

Some History

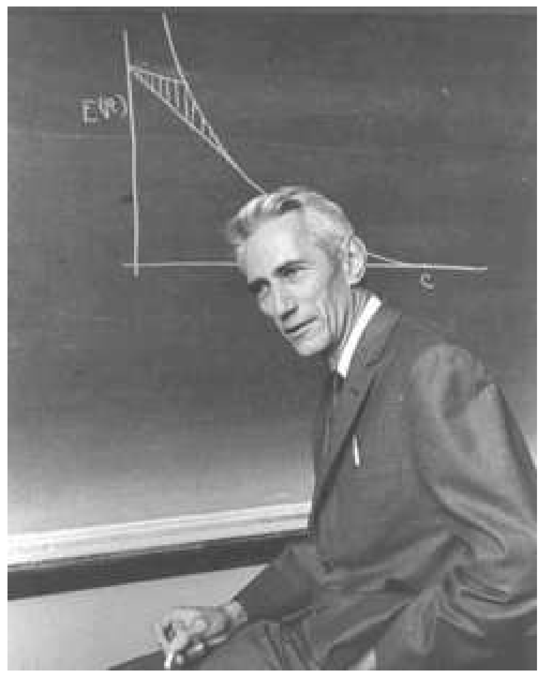
- Unlike most disciplines, the exact birth-date of information theory is known.
- C.E. Shannon, "A Mathematical Theory of Communication", Bell System Technical Journal, vol. 27, pp. 379-423, 623-656, July, October, 1948

Claude Shannon in 1948 (32 years old)

- Shannon proposed *both* a new problem and a few answers.

Claude Shannon, April 30, 1916 February 24, 2001

- Groundbreaking paper in 1937 as master student, *A Symbolic Analysis of Relay and Switching Circuits*, Transactions of IEEE, 1938.
- After graduate studies at MIT, Work at
  - Princeton, (Von Neumann, Einstein)
  - Bell labs, (Turing during war) mainly work on cryptography
  - back to MIT from 50's



- Closer to us, first recipient of the Kyoto prize in 1985,

Shannon's framework

- this diagram, from the original paper, defines the usual problems of communication

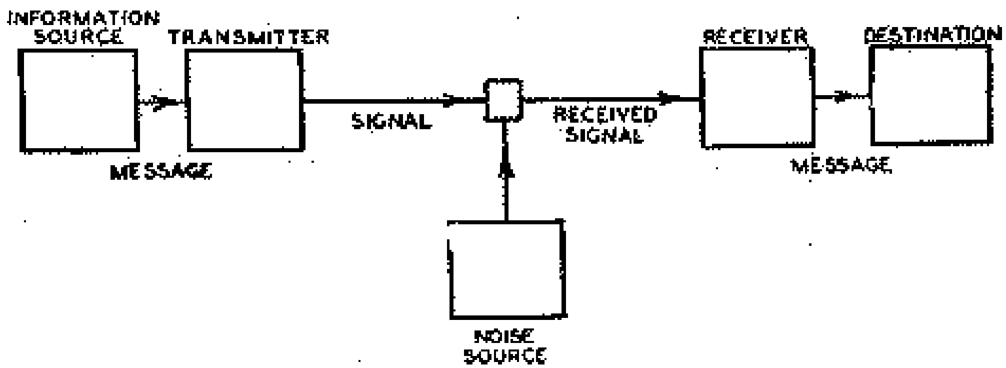


Fig. 1—Schematic diagram of a general communication system.

  - how to convert efficiently a message into a signal (transmitter)
  - how to decipher efficiently the signal back into a message (receiver)
  - how to cope with noisy environments which alter the signal.

- before Shannon, different approaches for each type of signal
  - telegraph,
  - texts,
  - codes,

- after Shannon, a unifying theory on all information.

A short movie by Charles and Ray Eames

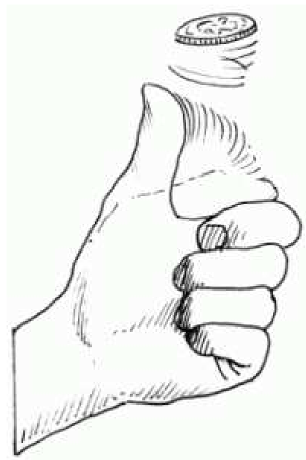
- The Eames couple are most known for their industrial design

- this documentary was shot in 1953...

- Merely 5 years after Shannon's breakthrough!

Shannon's framework through examples

- Example: you do N coin flips,



$\times N$

and record the results in a long word

$$b_1 \cdots b_N$$

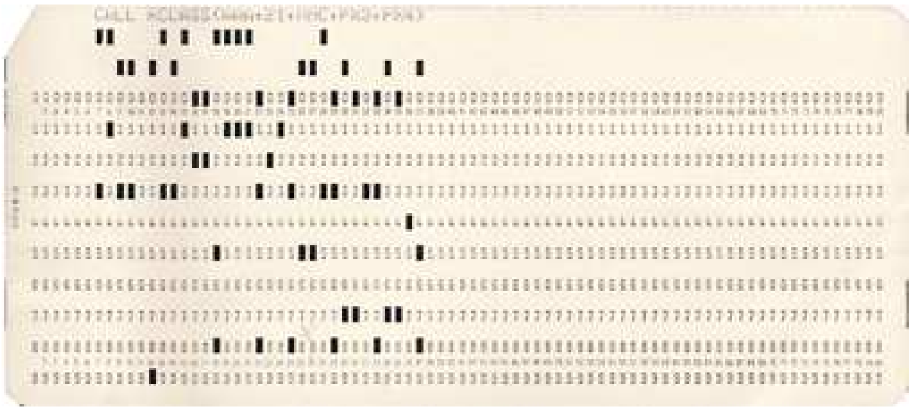
where each b_i is either **T**ails or **H**eads, that is $b_i \in \{\textbf{T,H}\}$.

- To keep things a bit more simple we use $\{0, 1\}$ instead of $\{\textbf{T,H}\}$.
- You want to send the outcome of this experiment of N coin tosses to someone.

Information and Entropy

- if $N = 1000$ you can write $0111011100 \cdots 101$ on a piece of paper and send it
- More handy approach: punch holes in cards,

  ...with the convention "hole=1", "no hole=0".



- to each hole corresponds one coin toss, ordered by time.
- The information given by each location (hole/no hole) on the card is a **bit**.

bit $=$ **bi**nary digi**t**, coined down in 1948 by Shannon (originally Tukey in '37)

Information and Entropy

- if $N = 1,000,000$, is there an **efficient** way to transmit this information?
- intuitively, this is hopeless:
  - each coin toss is **independent**,
  - each coin toss has two **equally likely outcomes, 1 or 0**.
  - you must provide the information for **each** coin toss.
- If your coin tosses is very atypical... $e.g.$

```
"I made 1,000,000 coin tosses and only had Heads"
```

...you may get away with a *very short* message...

- unfortunately, you will **more likely** need $1,000,000$ bits of information.
- we will show this later.

Information and Entropy

Suppose the coin is actually **biased**

- Suppose that the probability of tails (0) is $p_0 = 1/3$ & heads (1) is $p_1 = 2/3$.
- Yet we know that, on average, we will have to punch more holes than not, as

$$p_1 = 2/3 > p_0.$$

... *twice more 1's than 0's... we might consider punching 0's instead!!*

- yet, if we send the exact result, $b_1 b_2 \cdots b_N$, we still need $1,000,000$ bits.

What about **taking advantage** of the **differences** in probabilities $p_0 \neq p_1$ to **design a shorter message**?

Information and Entropy

- Simple approach: since the events are **independent**...

  ...for all $i \leq N-1$, the probability that

$$\begin{cases} p(b_i b_{i+1} = 00) = 1/9 \\ p(b_i b_{i+1} = 01) = 2/9 \\ p(b_i b_{i+1} = 10) = 2/9 \\ p(b_i b_{i+1} = 11) = 4/9 \end{cases}$$

- We could also consider 8 **triplets**, 16**quadruplets**, *etc.*
- Let's rewrite our tosses $b_1 \cdots b_N$ two by two, using some notations:
  - $a = 00, b = 01, c = 10, d = 11.$
  - We could send sequences of one of four letters, $abdcabb \cdots$.
- no gain so far... each letter needs $500.000\times$ 2 bits.

Information and Entropy

- remember that

$$\begin{cases} p(b_i b_{i+1} = a) = 1/9 \\ p(b_i b_{i+1} = b) = 2/9 \\ p(b_i b_{i+1} = c) = 2/9 \\ p(b_i b_{i+1} = d) = 4/9 \end{cases}$$

- Following the same idea, we will have, on average, a lot more d's than b or c's and few a's.
- Let's translate back a, b, c, d back into binary codes. Setting **for instance**[1]
  - $d = 0$,
  - $c = 10$,
  - $b = 110$,
  - $a = 111$.
- Intuition: **LONG** codewords for unlikely tokens.

[1] This is called a Huffman code

Information and Entropy

- In our example,

$$10\mathbf{11}10\mathbf{10}01\mathbf{11}01\mathbf{11} \quad (16\,\text{bits})$$
$$\Downarrow$$
$$cdccbdbd \quad (2\times\ \text{bits})$$
$$\Downarrow$$
$$10\mathbf{0}10\mathbf{10}110\mathbf{0}110\mathbf{0} \quad (\mathbf{15}\,\text{bits})$$

Information and Entropy

- **On average**, as N goes to infinity and given N tosses,
  - the naive technique needs N bits,
  - our trick requires $\frac{N}{2} \times (1p_d + 2p_c + 3p_b + 3p_d) = \frac{N}{2}(\frac{4}{9} + \frac{4}{9} + \frac{6}{9} + \frac{3}{9}) = \frac{17}{18}N$
- not so bad for such a simple trick.

- We could actually take advantage further of this trick by considering triplets, quadruplets etc...

- Shannon's theorem tells us something far more powerful

Information and Entropy

- For a random variable X taking values in a finite set $\mathcal{X}$ with probability p, we call the entropy of X,

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log_2 p(x)$$

N i.i.d. random variables *each* with entropy $H(X)$
can be compressed into more than $NH(X)$ bits with negligible risk of information loss, as N tends to infinity;
but conversely, if they are **compressed into fewer** than $NH(X)$ bits it is virtually certain that information **will be lost**.

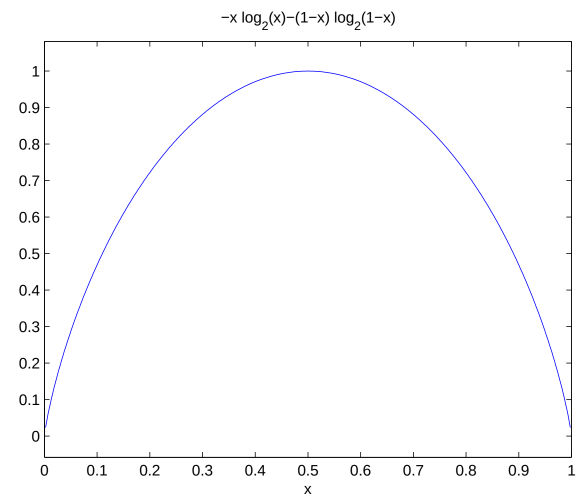
- In the previous example,

$$H(b) = -p_1 \log_2 p_1 - p_0 \log_2 p_0 = -\frac{2}{3} \log_2 \left(\frac{2}{3}\right) - \frac{1}{3} \log_2 \left(\frac{1}{3}\right) \approx 0.918$$

- We had $17/18 = 0.944...$ getting closer.

Entropy for binary random variables

- Two outcomes for a random variable X, 0 or 1.
- Two probabilities, $p_0 = p(X = 0)$ and $p_1 = p(X = 1)$.
- Moreover, $p_0 = p_1 - 1$, hence $H(X) = -p_1 \log p_1 - (1 - p_1) \log(1 - p_1)$.
- This is the curve represented below. $H(X) = 1$



- When $p_1 = \frac{1}{2}$, the entropy is at its **maximum**...

  ...*which is why we cannot do better, on average, than* ***actually*** *send* $1,000,000$ *bits if we want to* ***communicate*** $1,000,000$ *bits*...

Information and Entropy

> Whatever the method used to design the **signal**,
> if the word is made up of N **observations**
> of **i.i.d random variables** distributed like X,
> the **signal cannot be shorter on average than** $NH(X)$.

- Shannon's source code theorem gives a **lower bound**.
- The **reference length** becomes $NH(X)$,
- The main question of **coding and compression theory**:

> how to define **compression mechanisms (codes)**
> to **transform** messages into **shorter** signals
> so as to get **as close as possible to Shannon's bound**
> without necessarily knowing p?