Statistical Machine Learning
Assignment 1

This homework is due **October 22th (Wed.) 11:59 PM**

As you can see below, this homework only involves math questions.
You do not need to code this time.
Send your completed homework **in pdf format** to marcocuturicameto+report@gmail.com.
Please put the word `report` in the title of your email.

Positive Definite Matrices

A square $n \times n$ matrix A is positive definite if

$$\forall x \in \mathbb{R}^d, x \neq 0 \Rightarrow x^T X x > 0.$$

Alternatively, A is said to be positive *semi*-definite if

$$\forall x \in \mathbb{R}^d, x^T X x \geq 0.$$

1. Suppose A is a symmetric matrix. What can you say about its eigenvalues?
2. Suppose A is positive definite and symmetric. Prove that all the eigenvalues of A are positive. What can you say of these eigenvalues if A is a positive *semi*definite matrix?
3. Prove that the sum of two symmetric positive definite matrices $A, B \in \mathbb{R}^{d \times d}$ is positive definite.
4. Prove that if A is symmetric positive definite, then $\det A > 0$ and thus A is invertible. On the contrary, show that if $\det A > 0$, then A is not necessarily positive definite (you just need to provide a counterexample).
5. Prove that if A is positive *semi*definite and $\lambda > 0$, then $(A + \lambda I)$ is positive definite.
6. Prove that if $X \in \mathbb{R}^{d \times n}$ then XX^T and $X^T X$ are both positive semidefinite.
7. Prove that if $X \in \mathbb{R}^{d \times n}$ has rank d, then XX^T is positive definite (invertible).
8. Let $X \in \mathbb{R}^{d \times n}$ be a matrix, and $Y \in \mathbb{R}^n$. Prove that $\min_{\alpha \in \mathbb{R}^d} \|X^T \alpha - Y\|_2^2 + \lambda \|\alpha\|_2^2$ is attained for $\alpha = (XX^T + \lambda I)^{-1} XY$.
9. Compare this formula with the formula provided in Lecture 1. What is the advantage of introducing a positive λ parameter in the optimization above?