ORF 522

Linear Programming and Convex Analysis

Farkas lemma, dual simplex and sensitivity analysis

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Reminder

Covered duality theory in the general case.

- Lagrangian $L(x, \lambda, \mu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x)$
- Lagrange dual function $g(\lambda, \mu) = \inf_{x \in D} (f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x))$
- Lagrange dual function at any $\mu, \lambda \geq 0$ gives lower bounds for a min. problem.
- However, for most $\lambda, \mu$, the bound is $-\infty$.
- If we look for the optimum, we have a concave maximization problem.
- Always weak $(d^* < p^*)$ duality. Strong $(d^* < p^*)$ for some problems.

Looked more particularly at duality for LP’s.

- duals of LP’s are LP’s. LP’s are self-dual.
- Always strong duality.
- Complementary Slackness $u_i = v_j = 0$. 
Today

- Network flow example: Max-flow / Min-Cut.
- Strong Duality in LP’s through Farkas Lemma
- Strong duality illustration: gravity
- Dual Simplex
- Sensitivity Analysis... many case-scenarios for perturbation.
Network flow: Max-flow / Min-cut
Network flow: Max-flow / Min-cut

- $m$ nodes, $N_1, \cdots, N_m$.
- $d$ directed edges (arrows) to connect pairs of nodes $(N_i, N_i')$ in a set $\mathcal{V}$
  - Each edge carries a flow $f_k \geq 0$.
  - Each edge has a bounded capacity (pipe width) $f_k \leq u_k$.
- Relating edges and nodes: the network's incidence matrix $A \in \{−1, 0, 1\}^{m \times d}$:
  
  \[ A_{ik} = \begin{cases} 
  1 & \text{if edge } k \text{ starts at node } i \\
  -1 & \text{if edge } k \text{ ends at node } i \\
  0 & \text{otherwise} 
  \end{cases} \]

- For a node $i$,

  \[ \sum_{k \text{ s.t. edge ends at } i} f_k = \sum_{k \text{ s.t. edge starts at } i} f_k \]

- In matrix form: $Af = 0$
First problem: Maximal Flow

- We consider a **constant flow** from node 1 to node \( m \).
- What is the **maximal flow** that can go through the system?
- We **close the loop** with an **artificial edge** \((N_1, N_m)\), the \( d + 1 \)th edge.
- if \( u_{d+1} = \infty \), what would be the maximal flow \( f_{d+1} \) of that edge?
- Namely solve

  \[
  \begin{align*}
  \text{minimize} & \quad c^T f = -f_{d+1}, \\
  \text{subject to} & \quad [A , e] f = 0, \\
 & \quad 0 \leq f_1 \leq u_1, \\
 & \quad \vdots \\
 & \quad 0 \leq f_d \leq u_d, \\
 & \quad 0 \leq f_{d+1} \leq u_{d+1},
  \end{align*}
  \]

  with \( e = (-1, 0, \ldots, 0, 1) \) and \( c = (0, \ldots, 0, -1) \) and \( u_{d+1} \) a very large capacity for \( f_{d+1} \).
Second problem: Minimal Cut

- Suppose you are a **plumber** and you want to completely **stop the flow** from node $N_1$ to $N_m$.

- You have to remove **edges** (pipes). What is the minimal capacity you need to remove to **completely** stop the flow between $N_1$ to $N_m$?

- Goal: cut the set of nodes into two disjoint sets $S$ and $T$.

- Remove a set $C \subset V$ of edges and minimize the total capacity of $C$.

- $y_{i,j} \in \{0, 1\}$ will keep track of cuts. 1 for a cut, 0 otherwise.

- For each node $N_i$ there is a variable $z_i$ which is 0 if $N_i$ is in the set $S$ or 1 in the set $T$. We arbitrarily set $z_1 = 0$ and $z_N = 1$.

\[
\begin{align*}
\text{minimize} & \quad \sum_{(i,j)\in V} y_{i,j} w_{i,j} \\
\text{subject to} & \quad y_{i,j} + z_i - z_j \geq 0 \\
& \quad z_1 = 1, z_t = 0, z_i \geq 0, \\
& \quad y_{i,j} \geq 0, (i, j) \in V
\end{align*}
\]
Duality: example

- Let us form the **Lagrangian** of the Max-Flow problem:

\[
L(f, y, z) = c^T f + z^T [Ae] f + y^T (f - u)
\]

for \( f \geq 0 \) here.

- The **Lagrange dual function** is defined as

\[
g(y, z) = \inf_{f \geq 0} L(f, y, z)
= \inf_{f \geq 0} f^T \left( c + y + \begin{bmatrix} A^T \\ e^T \end{bmatrix} z \right) - u^T y
\]

- As usual, this infimum yields either \(-\infty\) or \(-u^T y\):

\[
g(y, z) = \begin{cases} 
-u^T y & \text{if } \left( c + y + \begin{bmatrix} A^T \\ e^T \end{bmatrix} z \right) \geq 0 \\
-\infty & \text{otherwise}
\end{cases}
\]
Duality: example

This means that the dual of the maximum flow problem is written:

\[
\begin{align*}
\text{minimize} & \quad u^T y \\
\text{subject to} & \quad c + y + \begin{bmatrix} A^T \\ e \end{bmatrix} z \geq 0
\end{align*}
\]

Compare the following dual with changed notations, from \(d + 1\) edges to \((d + 1)\) couple of points \((i, j) \in \mathcal{V}\)

\[
\begin{align*}
\text{minimize} & \quad \sum_{(i,j) \in \mathcal{V}} y_{ij} u_{ij} \\
\text{subject to} & \quad y_{N,1} + z_N - z_1 \geq 1 \\
& \quad y_{ij} + z_i - z_j \geq 0, \quad (i,j) \in \mathcal{V}, \\
& \quad y_{ij} \geq 0
\end{align*}
\]

to the minimum cut problem. The two problems are identical.
Duality: example

- The objective is to minimize:

\[ \sum_{(i,j) \in \mathcal{V}} u_{ij} y_{ij}, \quad (y_{i,j} \geq 0), \]

where \( u_{d+1} = u_{N,1} = M \) (very large), which means \( y_{N,1} = 0 \).

- The first equation then becomes:

\[ z_N - z_1 \geq 1 \]

so we can fix \( z_N = 1 \) and \( z_1 = 0 \).
Duality: example

- The equations for all the edges starting from $z_1 = 0$:

$$y_{1j} - z_j \geq 0$$

- Then, two scenarios are possible (no proof here):
  - $y_{1j} = 1$ with $z_j = 1$ and all the following $z_k$ will be ones in the next equations (at the minimum cost):

$$y_{jk} + z_j - z_k \geq 0, \quad (j, k) \in V$$

  - $y_{1j} = 0$ with $z_j = 0$ and we get the same equation for the next node:

$$y_{jk} - z_k \geq 0, \quad (j, k) \in V$$
Interpretation?

- If a node has $z_i = 0$, all the nodes preceding it in the network must have $z_j = 0$.
- If a node has $z_i = 1$, all the following nodes in the network must have $z_j = 1$.
- This means that $z_j$ effectively splits the network in two partitions.
- The equations:
  \[
  y_{ij} - z_i + z_j \geq 0
  \]
  mean for any two nodes with $z_i = 0$ and $z_j = 1$, we must have $y_{ij} = 1$.
- The objective minimizes the total capacity of these edges, which is also the capacity of the cut.
Strong duality and geometric interpretations
Proof for strong duality

- Remember the proof strategy:
  - We considered a **standard form** minimization first.
  - We used the **simplex** algorithm to reach a solution \( \mathbf{I} \).
  - The **reduced cost coefficient** at the optimum satisfies \( c^T - c^T_B I \mathbf{A} \geq 0 \).
  - We saw that writing \( \mu^T = c^T_B I \mathbf{A}^{-1} \) yielded a **feasible dual solution**.
  - That dual solution was furthermore **optimal** and shared the same objective with \( \mathbf{x}_I \).

- In the next slides,
  - We prove strong duality for LP’s through **Farkas’ Lemma**. No simplex argument.
  - We introduce a **physical analogy** often used to illustrated (strong) duality.
Farkas Lemma

- Basically states the feasibility of two **different** problems, two related problems.

**Theorem 1.** Let $A \in \mathbb{R}^{m \times n}$ and let $b \in \mathbb{R}^m$. Then exactly one of the two alternatives holds

1. there exists $x \geq 0$ such that $Ax = b$.
2. there exists $\mu$ such that $\mu^T A \geq 0$ and $\mu^T b < 0$. 

![Diagram of Farkas Lemma](image)
Farkas Lemma: Proof

• if (1), then suppose $\mu^T A \geq 0$. Through the solution $x$ of (1) we obtain

$$\mu^T A x = \mu^T b \geq 0,$$

which shows that (2) cannot be true.

• Let $S$ be the image of $A$ on $\mathbb{R}^n_+$, that is $S = \{Ax, x \geq 0\}$.
  
  o $S$ is convex, closed and contains 0.
  o If $b \notin S$, that is if (1) is not true, necessarily $\exists \mu$ such that $H_{\mu, \mu^T b}$ strictly isolates $S$ and $S \subset H^+_{\mu, \mu^T b}$
  o Since $0 \in S$, $\mu^T b < 0$.
  o On the other hand, every $\mu^T a_i \geq 0$. If not,
    ▷ for a sufficiently big positive $M$, $\mu^T (Ma_i) < \mu^T b$
    ▷ Contradiction since $Ma_i \in S$
  o Hence $\mu^T A \geq 0$ and since $\mu^T b < 0$, (2) is ensured.
Corollary 1. Let $a_1, a_2, \cdots, a_n$ and $b$ be given vectors and suppose that any vector $\mu$ that satisfies $\mu^T a_i \geq 0$, $i = 1, \ldots, n$, must also satisfy $\mu^T b \geq 0$. Then $b$ can be expressed as a nonnegative linear combination of the vectors $a_1, \cdots, a_n$.

- the first part of the sentence is the negation of (2) in the original Farkas lemma. Then necessarily (1) is true.
Proving Strong Duality with Farkas Lemma

- We have proved the following theorem:

**Theorem 2.** if an LP has an optima, so does its dual, and their respective optimal objectives are equal.

- Alternative proof: consider the primal-dual problems:

  \[
  \begin{align*}
  \text{minimize} & \quad c^T x \\
  \text{subject to} & \quad Ax \geq b \\
  \Rightarrow \quad & \\
  \text{maximize} & \quad b^T \mu \\
  \text{subject to} & \quad A^T \mu = c \\
  & \quad \mu \geq 0
  \end{align*}
  \]

- Let \( x^* \) be the primal optimal solution. Let us show \( \exists \mu^* \) dual solution with same cost.

  - \( J = \{ i | A^T_i x^* = b_i \} \) and let \( d \) be such that \( A^T_i d \geq 0 \) for \( i \in J \).
  
  - Consider \( \hat{x} = x^* + \varepsilon d \). We have \( A^T_i \hat{x} \geq A^T_i x^* = b_i \). feasible for \( J \)
  
  - For \( i \notin J \), \( A^T_i x^* > b_i \) and hence \( \hat{x} \) is feasible for \( \varepsilon \) sufficiently small. feasible
  
  - By optimality of \( x^* \) as a minimum, \( c^T x^* \leq c^T \hat{x} \) and \( c^T d \) must be nonnegative.
Proving Strong Duality with Farkas Lemma

- Through \( x^* \)'s optimality we have proved \( A_i^T d \geq 0 \) for \( i \in J \Rightarrow c^T d \geq 0 \).
- Using Farkas’ Lemma’s corollary, there must be \( \mu_i \geq 0, i \in J \) such that

\[
c = \sum_{i \in J} \mu_i A_i.
\]

- For \( i \notin J \) set \( \mu_i = 0 \).
- Thus \( \mu \geq 0 \) and \( \mu \) is dual feasible. Finally

\[
\mu^T b = \sum_{i \in J} \mu_i b_i = \sum_{i \in J} \mu_i A_i^T x^* = c^T x^*.
\]

- Through weak duality’s second corollary (primal and dual pair have same objective then both are optimal) we obtain strong duality.
• We have proved that Farkas’ lemma, a consequence of the isolation theorem, can prove strong duality.

• We follow with a widely used geometric and physical illustration of strong duality.

• Suppose we are in $\mathbb{R}^2$. We define a set of $m$ inequalities $A_i^T x \geq b_i$.

• A ball is thrown in the feasible set. Gravity makes it roll down to the lowest corner of the polyhedron.

• When in contact with the ball, each wall $i$ exerts a force $\mu_i A_i$ on the ball that is parallel to $A_i$. 
Gravity Example

- the position $x$ of the ball is the solution of

$$\begin{align*}
&\text{minimize} \quad c^T x \\
&\text{subject to} \quad A_i^T x \geq b_i, \quad i = 1..m,
\end{align*}$$

where $c$ points upwards, that is the opposite of the gravity vector.
Gravity Example

- The different walls exert forces $\mu_1 A_1, \mu_2 A_2, \cdots, \mu_m A_m$ on the ball. $\mu_i \geq 0$
- When $\mathbf{x}$ does not rest on wall $i$, $\mu_i = 0$ necessarily. Hence $\mu_i (b_i - A_i^T \mathbf{x}) = 0$.
- At the optimum, the forces cancel gravity: $\sum_{i=1}^m \mu_i A_i = \mathbf{c}$.
- At the optimum, $\mu^T \mathbf{b} = \sum_{i=1}^m \mu_i b_i = \sum_{i=1}^m \mu_i A_i^T \mathbf{x}^* = \mathbf{c}^T \mathbf{x} \Rightarrow \text{Strong duality}$
Dual Simplex Method
Intuition

- Strong duality proof using the simplex:
  - We start with a **standard form minimization**.
  - The **reduced cost coefficient** at the optimum satisfies \( c^T - c^T B^{-1}_I A \geq 0 \).
  - We saw that writing \( \mu^T = c^T B^{-1}_I \) yielded a **feasible dual solution**.
  - That dual solution was **optimal** and shared the same objective with \( x_I \).

- There is some **obvious symmetry** between the **reduced cost coefficient** and the **solution** for a given base.
Primal and dual simplex in a few words:

- Given a BFS for the primal, the **primal** simplex looks for a **dual feasible solution** $\mu^T = c_I^T B_I^{-1}$ while maintaining **primal feasibility** for $x$.

- Given a dual BFS, the **dual** simplex looks for a **primal feasible solution** $x$ while maintaining **dual feasibility** for $\mu$.

Why consider it? great for understanding. useful for sensitivity analysis.
A not so distant reminder on tableaux

\[
\begin{array}{ccc}
\vdots & \cdots & \vdots \\
\vdots & B_I^{-1}A & \vdots \\
\vdots & \vdots & \vdots \\
\cdots & (c - c_I^T B_I^{-1} A)^T & \cdots \\
\end{array}
\quad
\begin{array}{c}
\vdots \\
B_I^{-1}b \\
\vdots \\
-c_I^T B_I^{-1} b \\
\end{array}
\]

In the dual simplex iterations,

- we do not assume that $B_I^{-1}b$ is nonnegative at each iteration.
- we assume that $(c - c_I^T B_I^{-1} A)^T \geq 0$, or equivalently that $\mu^T A \leq c^T$.

This means $\mu = B_I^{-1}c_I$ is dual-feasible...

Note the analogy between $c_I^T B_I^{-1}$ or $B_I^{-1}c_I$ and $B_I^{-1}b$.

If by any chance both $(c - c_I^T B_I^{-1} A)^T \geq 0$ and $B_I^{-1} b \geq 0$ then we have found the solution.

If not... basis change!
Pivot

• Let’s write $r = c - c^T B^{-1}_I A$.

• Select a **primal** variable $i_l$ s.t. $(B^{-1}_I b)_l < 0$ and consider the tableau $l$th row.

• That row is made of $(y_{l,i})_{1 \leq i \leq n}$ coordinates.
  
  ○ for each $i$ such that $y_{l,i} < 0$, consider the ratio $\frac{r_i}{|y_{l,i}|}$,
  
  ○ let $j$ be the column number for which this ratio is smallest.
  
  ○ $j$ must correspond to a nonbasic variable (otherwise $y_{l,j}$ is zero or 1 for $y_{l,i}$).
  
  ○ Then completely standard pivot on $y_{l,j}$: $I \leftarrow I \setminus \{i_l\} \cup \{j\}$.
  
  ○ Can prove that the new reduced cost coefficients stay positive, and we keep dual-feasibility.
Dual Simplex Pivot Example

- The following tableau is dual feasible

\[
\begin{array}{cccccc}
-2 & 4 & 1 & 1 & 0 & 2 \\
4 & -2 & -3 & 0 & 1 & -1 \\
2 & 6 & 10 & 0 & 0 & 0 \\
\end{array}
\]

- The basis \( I = \{4, 5\} \). The current solutions’ second variable \((B^{-1}_I b)_2\) is negative.

- Negative entries for the second row can be found in 2nd and 3rd variables

- Corresponding ratios \(6/|−2|\) and \(10/|−3|\). Therefore \(I’ = \{4, 5\} \setminus \{5\} \cup \{2\}\) and we pivot accordingly

\[
\begin{array}{ccccc}
6 & 0 & -5 & 1 & 2 & 0 \\
-2 & 1 & 3/2 & 0 & -1/2 & 1/2 \\
14 & 0 & 1 & 0 & 3 & -3 \\
\end{array}
\]

- primal and dual feasible... optimal and optimum is 3
The dual simplex proceeds in the same way that the (primal simplex)
Any base $I$, defines a primal $B_I^{-1}b$ and dual solution $(c_I^T B_I^{-1})A \leq c^T$.
Assume $I$ provides a dual feasible solution.
Update the base through two criterions:
  - The column $B_I^{-1}b$ has negative elements? that gives exiting index $i_l$.
  - Is there a pivot feasible for the reduced costs? entering column $j$.
Pivot and update the whole tableau.
Dual Simplex Summary

● When can/should we use the dual simplex?
  ○ We have a base $I$ that is dual-feasible to start our problem.
  ○ We have a solution $x^*$ with base $I$ for a problem and we only change the constraints $b$. 
Sensitivity Analysis
Sensitivity analysis

Let’s study sensitivity with a generic problem and its dual:

\[
\begin{align*}
\text{minimize} & \quad f_0(x) & & \text{maximize} & \quad g(\lambda, \mu) \\
\text{subject to} & \quad f_i(x) \leq 0, & & \text{subject to} & \quad \lambda \geq 0, \\
& \quad h_i(x) = 0, & & & \lambda = 1, \ldots, m \\
& & & & i = 1, \ldots, p \\
\end{align*}
\]

Consider a small perturbation \((u, v)\) to the constraints:

\[
\begin{align*}
\text{minimize} & \quad f_0(x) & & \text{maximize} & \quad g(\lambda, \mu) - \lambda^T u - \mu^T v \\
\text{subject to} & \quad f_i(x) \leq u_i, & & \text{subject to} & \quad \lambda \geq 0, \\
& \quad h_i(x) = v_i, & & & i = 1, \ldots, m \\
& & & & i = 1, \ldots, p \\
\end{align*}
\]

Here \(x, \lambda, \mu\) are variables and \((u, v)\) parameters.

We write \(p^*(u, v)\) for the optimum of the problem given perturbations \(u, v\).

This value may not be defined if the problem is unfeasible...
Global sensitivity analysis

- Suppose we have **strong duality** in the original problem, i.e. \( \exists \lambda^* \geq 0, \mu^* \) s.t. \( P^*(0, 0) = g(\lambda^*, \mu^*) \).

- For \((u, v)\) such that \( p^*(u, v) \) is defined, by weak duality,

\[
p^*(u, v) \geq g(\lambda^*, \nu^*) - u^T \lambda^* - v^T \mu^* \\
\geq p^*(0, 0) - u^T \lambda^* - v^T \mu^*
\]

- This gives a global lower bound, and indications on \( p^* \) for some changes:
  - If \( \lambda_i^* \gg 0, u_i < 0 \) (tighten constraint), then **big increase** for \( p^* \).
  - If \( \lambda_i^* \) is small, \( u_i > 0 \) (loosen constraint), then **little impact** on \( p^* \).
  - If \( \mu_i^* \gg 0 \) and \( v_i < 0 \) or \( \mu_i^* \ll 0 \) and \( v_i > 0 \) then **big increase** for \( p^* \).
  - If \( \mu_i^* \approx 0^+ \) and \( v_i > 0 \) or \( \mu_i^* \approx 0^- \) and \( v_i < 0 \) then **little impact** on \( p^* \).
Local sensitivity analysis

• Suppose $p^*$ is differentiable around $u = 0, v = 0$.

• Hence, for small values $(u, v)$ we have:

$$
\lambda_i^* = -\frac{\partial p^*(0, 0)}{\partial u_i}, \quad \mu_i^* = -\frac{\partial p^*(0, 0)}{\partial v_i}
$$

• The dual solution gives the local **sensitivities** of the optimal objective with respect to constraint perturbations.

• This time the interpretation is **symmetric**.

• The objective moves by $-\lambda_i^* u_i$ whatever the signs of $\lambda_i^*$ and $u_i$. 
Sensitivity Analysis, The LP case
Sensitivity analysis, the LP case

- Suppose we have a standard form LP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

- As usual, assume \( I \) is the optimal base and \( x^* \) the optimum.

- Suppose \( b \) is replaced by \( b + d \) where \( d \approx 0 \).
  - As long as \( x^* \) is non-degenerate and \( d \) small, \( B_I^{-1}(b + d) \geq 0 \). feasible
  - Since \( I \) is optimal, \( c - c_I^T B_I^{-1} A \geq 0 \). still optimal

- Hence the same basis is still optimal for an infinitesimally perturbed problem.
Sensitivity analysis, the LP case

• The new optimum is

$$c^T_i B_i^{-1} (b + d) = \mu^T (b + d)$$

• perturbation $d$: $z^*$ becomes $z^* + \mu^T d$.

• each component $\mu_i$ can be interpreted as the marginal cost of each unit increase of $b_i$.

• Such marginal costs are also called shadow prices.
Sensitivity: examples

- The **simplex** can handle more *advanced perturbation scenarios*.
- Suppose we have converged to an optimum $\mathbf{I}$ and have access to $\mathbf{x}^*$ and $\mathbf{\mu}^*$.
- We review the following scenarios:
  1. A new variable is added
  2. A new inequality constraint is added
  3. A new equality constraint is added
  4. The constraint vector $\mathbf{b}$ is changed
  5. The cost vector $\mathbf{c}$ is changed
  6. A nonbasic column of $\mathbf{A}$ changes
  7. A basic columns of $\mathbf{A}$ changes

  and discuss how we can still use $\mathbf{I}$ to get the new optimum quickly.
1. New variable

- Suppose the program becomes

\[
\begin{align*}
\text{minimize} & \quad c^T x + c_{n+1} x_{n+1} \\
\text{subject to} & \quad A x + a_{n+1} x_{n+1} = b \\
& \quad x \geq 0, x_{n+1} \geq 0
\end{align*}
\]

- Note that \((x^*, 0)\) is already a BFS of the new problem.

- For the basis \(I\) to remain optimal, we need that

\[
\begin{align*}
c_{n+1} - c_I^T B_I^{-1} a_{n+1} \geq 0.
\end{align*}
\]

- If this is the case, \(I\) is still optimal.

- If not, we start from \((x^*, 0)\) and use the simplex algorithm.

- Running time typically much lower than rerunning everything from scratch.
2. New inequality

- Suppose the program has a new constraint $A_{m+1}^T x \geq b_{m+1}$.
- If $x^*$ already satisfies this inequality, then $x^*$ is still optimal.
- If not, introduce a surplus variable $x_{n+1}$ and $A_{m+1}^T x - x_{n+1} = b_{m+1}$.
- We obtain the following standard form, writing $\beta = \begin{bmatrix} b \\ b_{m+1} \end{bmatrix}$ and $x \in \mathbb{R}^{n+1}$,

\[
\begin{array}{l}
\text{minimize} & c^T x \\
\text{subject to} & \begin{bmatrix} A_{m+1}^T & 0 \\ A_{m+1}^T & -1 \end{bmatrix} x = \beta \\
& x \geq 0
\end{array}
\]
2. New inequality

- We use a basis $\mathbf{I}' = \mathbf{I} \cup \{n + 1\}$. Write $\mathbf{a} = A_{m+1, \mathbf{I}}$ Note that

$$B_{\mathbf{I}'} = \begin{bmatrix} B_{\mathbf{I}} & 0 \\ \mathbf{a}^T & -1 \end{bmatrix}, \quad \det B_{\mathbf{I}'} = -\det B_{\mathbf{I}} \neq 0, \quad B_{\mathbf{I}'}^{-1} = \begin{bmatrix} B_{\mathbf{I}}^{-1} & 0 \\ \mathbf{a}^T B_{\mathbf{I}}^{-1} & -1 \end{bmatrix}.$$ 

- The corresponding primal point is $[x^* \quad \mathbf{a}^T x^* - b_{m+1}]$. It is infeasible by assumption.

- On the other hand the new reduced cost is given by

$$[\mathbf{c}^T 0] - [\mathbf{c}_{\mathbf{I}}^T 0] \begin{bmatrix} B_{\mathbf{I}}^{-1} & 0 \\ \mathbf{a}^T B_{\mathbf{I}}^{-1} & -1 \end{bmatrix} \begin{bmatrix} A_{m+1} & 0 \\ -1 \end{bmatrix} = [\mathbf{c}^T - \mathbf{c}_{\mathbf{I}}^T B_{\mathbf{I}}^{-1} A \quad 0] \quad \text{which is thus nonnegative by optimality of } \mathbf{I}.$$ 

- Hence $\mathbf{I}'$ is dual feasible... dual simplex with the tableau given by $B_{\mathbf{I}'}^{-1}$.
3. New equality

- New constraint $A_{m+1}^T x = b_{m+1}$ and suppose $A_{m+1}^T x^* > b_{m+1}$
- The dual of the new problem becomes

  maximize $\mu^T b$

  subject to $\begin{bmatrix} \mu^T & \mu_{m+1} \end{bmatrix} \begin{bmatrix} A_{m+1}^T \\ A_{m+1} \end{bmatrix} \leq c^T$.

  where $\mu_{m+1}$ is a new dual variable associated with the latest constraint.
- If $\mu^*$ is the optimal dual solution for $I^*$, $(\mu^*, 0)$ is feasible, but we have no base $I$ that corresponds to $(\mu^*, 0)$...
- Back to the primal. We modify it by an auxiliary problem with $M \gg 0$

  minimize $c^T x + M x_{n+1}$

  subject to $Ax = b$

  $A_{m+1}^T x - x_{n+1} = b_{m+1}$

  $x, x_{m+1} \geq 0$

- We can then use the approach in (2) by considering $B_{I'} = \begin{bmatrix} B_I & 0 \\ a^T & -1 \end{bmatrix}$. 
4. Change in constraint vector $b$

- Suppose $b_j$ of $b$ is changed to $b_j + \delta$, that is $b$ is changed to $b + \delta e_j$.
- For what range of $\delta$ will $I$ remain feasible? remember that optimality is not affected.
- Let $B_I^{-1} = [\beta_{i,j}]$. The condition $B_I^{-1}(b + \delta e_j) \geq 0$ is equivalent to

$$\max_{\{i | \beta_{ij} > 0\}} -\frac{(B_I^{-1}b)_i}{\beta_{ij}} \leq \delta \leq \min_{\{i | \beta_{ij} < 0\}} -\frac{(B_I^{-1}b)_i}{\beta_{ij}}$$

- For this range, the optimal cost is given by $c_I^T B_I^{-1}(b + \delta e_j) = \mu_*^T b + \delta \mu_*^j$.
- Outside the range, run the dual simplex starting with $\mu^*$.
5. Change in cost vector $\mathbf{c}$

- Suppose $c_j$ of $\mathbf{c}$ is changed to $c_j + \delta$, that is $\mathbf{c}$ is changed to $\mathbf{c} + \delta \mathbf{e}_j$.
- Primal feasibility of $\mathbf{I}$ is not affected. However, we need to check $\mathbf{c}_I^T \mathbf{B}_I^{-1} \mathbf{A} \leq \mathbf{c}^T$.
- If $j$ corresponds to a **nonbasic** variable, $\mathbf{c}_I$ does not change, but we need that
  \[-(c_j - \mathbf{c}_I^T \mathbf{B}_I^{-1} \mathbf{a}_j) \leq \delta.
\]
  If this is not ensured, we have to apply a few primal simplex iterations.
- If $j$ corresponds to a **basic** variable, i.e. $i_l = j$, then the condition becomes
  \[(\mathbf{c}_I + \delta \mathbf{e}_l)^T \mathbf{B}_I^{-1} \mathbf{a}_i \leq c_i.
\]
- Equivalently, $\delta y_{l,i} \leq c_i - \mathbf{c}_I^T \mathbf{B}_I^{-1} \mathbf{a}_i$ ensures the solution remains optimal.
6. Change in nonbasic column of \( A \)

- The \( i \)th coordinate of a **nonbasic** column vector \( \mathbf{a}_j \) is changed to \( a_{ij} + \delta \).
- If the variable is nonbasic, primal feasibility is not affected.
- Dual feasibility: \( c_j - \mu^T(\mathbf{a}_j + \delta \mathbf{e}_i) \geq 0 \).
- If this inequality is violated, \( j \) can be inserted in the basis, requiring a primal simplex step.
7. Change in nonbasic column of $A$

- The $i$th coordinate of a **basic** column vector $a_j$ is changed to $a_{ij} + \delta$, both feasibility and optimality conditions are affected.

- exercise...
Next time

- Ellipsoid Method and Polynomial Complexity of the Simplex.