# ORF 522

# Linear Programming and Convex Analysis

## Farkas lemma, dual simplex and sensitivity analysis

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# Reminder

## Covered duality theory in the general case.

- Lagrangian $L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f_0(\boldsymbol{x}) + \sum_{i=1}^{m} \boldsymbol{\lambda_i} f_i(\boldsymbol{x}) + \sum_{i=1}^{p} \boldsymbol{\mu_i} h_i(\boldsymbol{x})$
- Lagrange dual function $g(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \inf_{x \in \mathcal{D}} \left( f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \mu_i h_i(x) \right)$
- Lagrange dual function at any $\mu, \lambda \geq 0$ gives **lower bounds** for a min. problem.
- However, **for most** $\lambda, \mu$, the bound is $-\infty$.
- If we look for the optimum, we have a **concave maximization** problem.
- **Always weak** $(d^\star < p^\star)$ duality. Strong $(d^\star < p^\star)$ for some problems.

## Looked more particularly at duality for LP's.

- duals of LP's are LP's. **LP's are self-dual**.
- **Always strong** duality.
- **Complementary Slackness** $u_i = v_j = 0$.

# Today

- Network flow example: Max-flow / Min-Cut.
- Strong Duality in LP's through Farkas Lemma
- Strong duality illustration: gravity
- Dual Simplex
- Sensitivity Analysis... many case-scenarios for perturbation.

# Network flow: Max-flow / Min-cut

# Network flow: Max-flow / Min-cut

- $m$ nodes, $N_1, \cdots, N_m$.
- $d$ directed edges (arrows) to connect pairs of nodes $(N_i, N_{i'})$ in a set $\mathcal{V}$
  - Each edge carries a flow $f_k \geq 0$.
  - Each edge has a bounded capacity (pipe width) $f_k \leq u_k$
- Relating edges and nodes: the network's incidence matrix $A \in \{-1, 0, 1\}^{m \times d}$:

$$A_{ik} = \begin{cases} 1 & \text{if edge } k \text{ starts at node } i \\ -1 & \text{if edge } k \text{ ends at node } i \\ 0 & \text{otherwise} \end{cases}$$

- For a node $i$,

$$\sum_{k \text{ s.t. edge ends at } i} f_k = \sum_{k \text{ s.t. edge starts at } i} f_k$$

- In matrix form: $A\mathbf{f} = \mathbf{0}$

# First problem: Maximal Flow

- We consider a **constant flow** from node $1$ to node $m$.
- What is the **maximal flow** that can go through the system?
- We **close the loop** with an *artificial edge* $(N_1, N_m)$, the $d+1$th edge.
- if $u_{d+1} = \infty$, what would be the maximal flow $f_{d+1}$ of that edge?
- Namely solve

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{f} = -f_{d+1}, \\ \text{subject to} & [A\ , e]\,\mathbf{f} = 0, \\ & 0 \leq f_1 \leq u_1, \\ & \quad\vdots \\ & 0 \leq f_d \leq u_d, \\ & 0 \leq f_{d+1} \leq u_{d+1}, \end{array}$$

with $\mathbf{e} = (-1, 0, \ldots, 0, 1)$ and $\mathbf{c} = (0, \ldots, 0, -1)$ and $u_{d+1}$ a very large capacity for $f_{d+1}$.

# Second problem: Minimal Cut

- Suppose you are a **plumber** and you want to completely **stop the flow** from node $N_1$ to $N_m$.
- You have to remove **edges** (pipes). What is the minimal capacity you need to remove to **completely** stop the flow between $N_1$ to $N_m$?
- Goal: cut the set of nodes into two disjoint sets $S$ and $T$.
- Remove a set $\mathcal{C} \subset \mathcal{V}$ of edges and minimize the total capacity of $\mathcal{C}$.
- $y_{ij} \in \{0, 1\}$ will keep track of cuts. 1 for a cut, 0 otherwise.
- For each node $N_i$ there is a variable $z_i$ which is 0 if $N_i$ is in the set $S$ or 1 in the set $T$. We arbitrarily set $z_1 = 0$ and $z_N = 1$.

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{(i,j)\in\mathcal{V}} y_{ij} u_{ij} \\ \text{subject to} & y_{i,j} + z_i - z_j \geq 0 \\ & z_1 = 1, z_t = 0, z_i \geq 0, \\ & y_{ij} \geq 0, (i,j) \in \mathcal{V} \end{array}$$

# Duality: example

- Let us form the **Lagrangian** of the Max-Flow problem:

$$L(\mathbf{f},\mathbf{y},\mathbf{z}) = \mathbf{c}^T\mathbf{f} + \mathbf{z}^T\,[Ae]\,\mathbf{f} + \mathbf{y}^T(\mathbf{f}-\mathbf{u})$$

  for $\mathbf{f} \geq 0$ here.

- The **Lagrange dual function** is defined as

$$\begin{aligned} g(\mathbf{y},\mathbf{z}) &= \inf_{\mathbf{f}\geq 0} L(\mathbf{f},\mathbf{y},\mathbf{z}) \\ &= \inf_{\mathbf{f}\geq 0} \mathbf{f}^T \left( \mathbf{c} + \mathbf{y} + \left[ \begin{array}{c} A^T \\ \mathbf{e}^T \end{array} \right] \mathbf{z} \right) - \mathbf{u}^T\mathbf{y} \end{aligned}$$

- As usual, this infimum yields either $-\infty$ or $-\mathbf{u}^T\mathbf{y}$:

$$g(\mathbf{y},\mathbf{z}) = \begin{cases} -\mathbf{u}^T\mathbf{y} & \text{if } \left( \mathbf{c} + \mathbf{y} + \left[ \begin{array}{c} A^T \\ \mathbf{e}^T \end{array} \right] \mathbf{z} \right) \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

# Duality: example

This means that the **dual** of the maximum flow problem is written:

$$\begin{array}{ll} \text{minimize} & \mathbf{u}^T\mathbf{y} \\ \text{subject to} & \mathbf{c} + \mathbf{y} + \begin{bmatrix} A^T \\ \mathbf{e} \end{bmatrix} \mathbf{z} \geq 0 \end{array}$$

Compare the following dual with changed notations, from $d+1$ edges to $(d+1)$ couple of points $(i,j) \in \mathcal{V}$

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j)\in\mathcal{V}} y_{ij} u_{ij} \\ \text{subject to} & y_{N,1} + z_N - z_1 \geq 1 \\ & y_{ij} + z_i - z_j \geq 0, \quad (i,j) \in \mathcal{V}, \\ & y_{ij} \geq 0 \end{array}$$

to the **minimum cut problem**. The two problems are **identical**.

# Duality: example

- The objective is to minimize:

$$\sum_{(i,j)\in\mathcal{V}} u_{ij}y_{ij}, \quad (y_{i,j} \geq 0),$$

  where $u_{d+1} = u_{N,1} = M$ (very large), which means $y_{N,1} = 0$.

- The first equation then becomes:

$$z_N - z_1 \geq 1$$

  so we can fix $z_N = 1$ and $z_1 = 0$.

# Duality: example

- The equations for all the edges starting from $z_1 = 0$:

$$y_{1j} - z_j \geq 0$$

- Then, two scenarios are possible (no proof here):
  - $y_{1j} = 1$ with $z_j = 1$ and all the following $z_k$ will be ones in the next equations (at the minimum cost):

$$y_{jk} + z_j - z_k \geq 0, \quad (j,k) \in \mathcal{V}$$

  - $y_{1j} = 0$ with $z_j = 0$ and we get the same equation for the next node:

$$y_{jk} - z_k \geq 0, \quad (j,k) \in \mathcal{V}$$

# Duality: example

Interpretation?

- If a node has $z_i = 0$, all the nodes preceding it in the network must have $z_j = 0$.
- If a node has $z_i = 1$, all the following nodes in the network must have $z_j = 1 \ldots$
- This means that $z_j$ effectively splits the network in two partitions
- The equations:
$$y_{ij} - z_i + z_j \geq 0$$
mean for any two nodes with $z_i = 0$ and $z_j = 1$, we must have $y_{ij} = 1$.
- The objective minimizes the total capacity of these edges, which is also the capacity of the cut.

# Strong duality and geometric interpretations

# Proof for strong duality

- Remember the proof strategy:
  - We considered a **standard form** *minimization* first.
  - We used the **simplex** algorithm to reach a solution $\mathbf{I}$.
  - The **reduced cost coefficient** at the optimum satisfies $c^T - c_{\mathbf{I}}^T B_{\mathbf{I}}^{-1} A \geq 0$.
  - We saw that writing $\mu^T = c_{\mathbf{I}}^T B_{\mathbf{I}}^{-1}$ yielded a **feasible dual solution**.
  - That dual solution was furthermore **optimal** and shared the same objective with $\mathbf{x}_{\mathbf{I}}$.

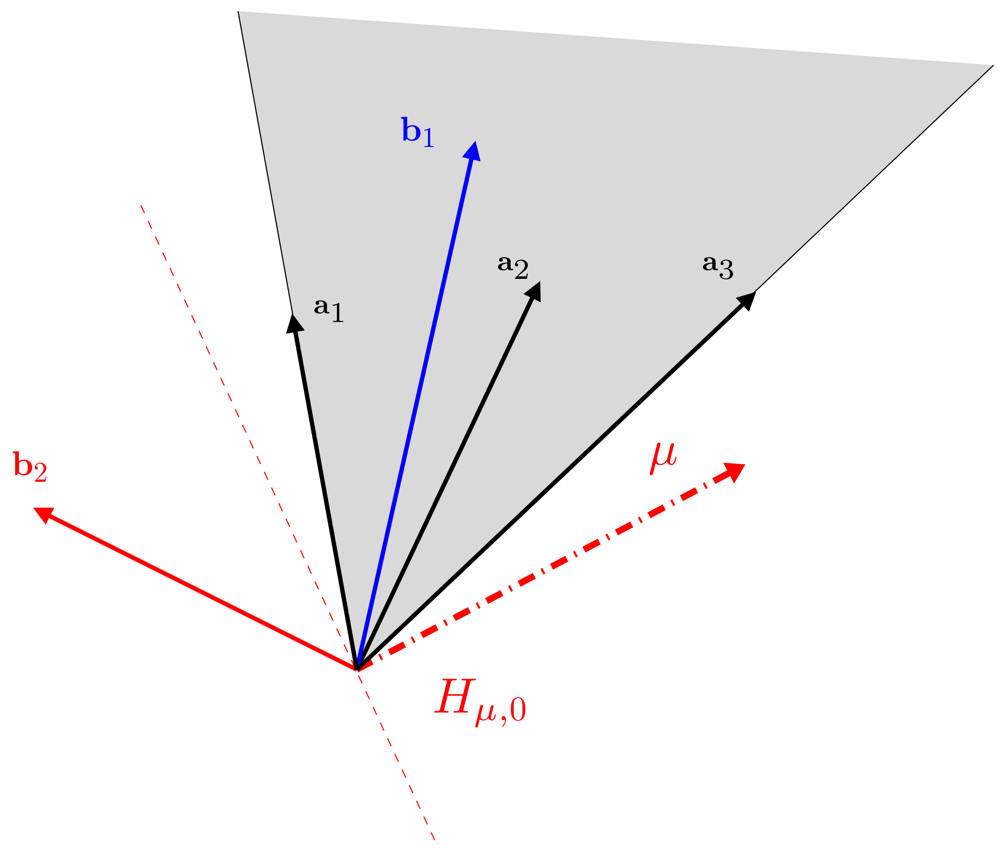
- In the next slides,
  - We prove strong duality for LP's through **Farkas' Lemma**. No simplex argument.
  - We introduce a **physical analogy** often used to illustrated (strong) duality.

# Farkas Lemma

- Basically states the feasibility of two **different** problems, two **related** problems.

**Theorem 1.** *Let* $A \in \mathbf{R}^{m \times n}$ *and let* $\mathbf{b} \in \mathbf{R}^m$*. Then exactly one of the two alternatives holds*

1. *there exists* $\mathbf{x} \geq \mathbf{0}$ *such that* $A\mathbf{x} = \mathbf{b}$.
2. *there exists* $\boldsymbol{\mu}$ *such that* $\boldsymbol{\mu}^T A \geq 0$ *and* $\boldsymbol{\mu}^T \mathbf{b} < 0$.



# Farkas Lemma: Proof

- if (1), then suppose $\mu^T A \geq 0$. Through the solution $\mathbf{x}$ of (1) we obtain

$$\mu^T A\mathbf{x} = \mu^T \mathbf{b} \geq 0,$$

  which shows that (2) cannot be true.

- Let $S$ be the image of $A$ on $\mathbf{R}_+^n$, that is $S = \{A\mathbf{x}, \mathbf{x} \geq 0\}$.
  - $S$ is convex, closed and contains $\mathbf{0}$.
  - If $\mathbf{b} \notin S$, that is if (1) is not true, necessarily $\exists \mu$ such that $H_{\mu,\mu^T\mathbf{b}}$ strictly isolates $S$ and $S \subset H^+_{\mu,\mu^T\mathbf{b}}$
  - Since $\mathbf{0} \in S$, $\mu^T \mathbf{b} < 0$.
  - On the other hand, every $\mu^T \mathbf{a}_i \geq 0$. If not,
    - ▷ for a sufficiently big positive $M$, $\mu^T(M\mathbf{a}_i) < \mu^T \mathbf{b}$
    - ▷ Contradiction since $M\mathbf{a}_i \in S$
  - Hence $\mu^T A \geq 0$ and since $\mu^T \mathbf{b} < 0$, (2) is ensured.

# Farkas Lemma: An immediate corollary

**Corollary 1.** *Let* $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ *and* $\mathbf{b}$ *be given vectors and suppose that any vector* $\mu$ *that satisfies* $\mu^T \mathbf{a}_i \geq 0$, $i = 1, .., n$, *must also satisfy* $\mu^T \mathbf{b} \geq 0$. *Then* $\mathbf{b}$ *can be expressed as a nonnegative linear combination of the vectors* $\mathbf{a}_1, \cdots, \mathbf{a}_n$.

- the first part of the sentence is the negation of (2) in the original Farkas lemma. Then necessarily (1) is true.

# Proving Strong Duality with Farkas Lemma

- We have proved the following theorem:

  **Theorem 2.** *if an LP has an optima, so does its dual, and their* ***respective optimal objectives are equal****.*

- Alternative proof: consider the primal-dual problems:

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \geq \mathbf{b} \end{array} \quad \Rightarrow \quad \begin{array}{ll} \text{maximize} & \mathbf{b}^T\mu \\ \text{subject to} & A^T\mu = \mathbf{c} \\ & \mu \geq 0 \end{array}$$

- Let $\mathbf{x}^\star$ be the primal optimal solution. Let us show $\exists\mu^\star$ dual solution with same cost.
  - $J = \{i | A_i^T\mathbf{x}^\star = b_i\}$ and let $\mathbf{d}$ be such that $A_i^T\mathbf{d} \geq 0$ for $i \in J$.
  - Consider $\hat{\mathbf{x}} = \mathbf{x}^\star + \varepsilon\mathbf{d}$. We have $A_i^T\hat{\mathbf{x}} \geq A_i^T\mathbf{x}^\star = b_i$. **feasible for $J$**
  - For $i \notin J$, $A_i^T\mathbf{x}^\star > b_i$ and hence $\hat{\mathbf{x}}$ is feasible for $\varepsilon$ sufficiently small. **feasible**
  - By **optimality** of $\mathbf{x}^\star$ as a minimum, $\mathbf{c}^T\mathbf{x}^\star \leq \mathbf{c}^T\tilde{\mathbf{x}}$ and $\mathbf{c}^T\mathbf{d}$ must be nonnegative.

# Proving Strong Duality with Farkas Lemma

- Through $\mathbf{x}^\star$'s **optimality** we have proved $A_i^T \mathbf{d} \geq 0$ for $i \in J \Rightarrow \; \mathbf{c}^T \mathbf{d} \geq 0$.
- Using **Farkas' Lemma's corollary**, there must be $\mu_i \geq 0, i \in J$ such that

$$\mathbf{c} = \sum_{i \in J} \mu_i A_i.$$

- For $i \notin J$ set $\mu_i = 0$.
- Thus $\mu \geq 0$ and $\mu$ is dual feasible. Finally

$$\mu^T \mathbf{b} = \sum_{i \in J} \mu_i b_i = \sum_{i \in J} \mu_i A_i^T \mathbf{x}^\star = \mathbf{c}^T \mathbf{x}^\star.$$

- Through weak duality's second corollary (*primal and dual pair have same objective then both are optimal*) we obtain **strong** duality.

# Gravity Example

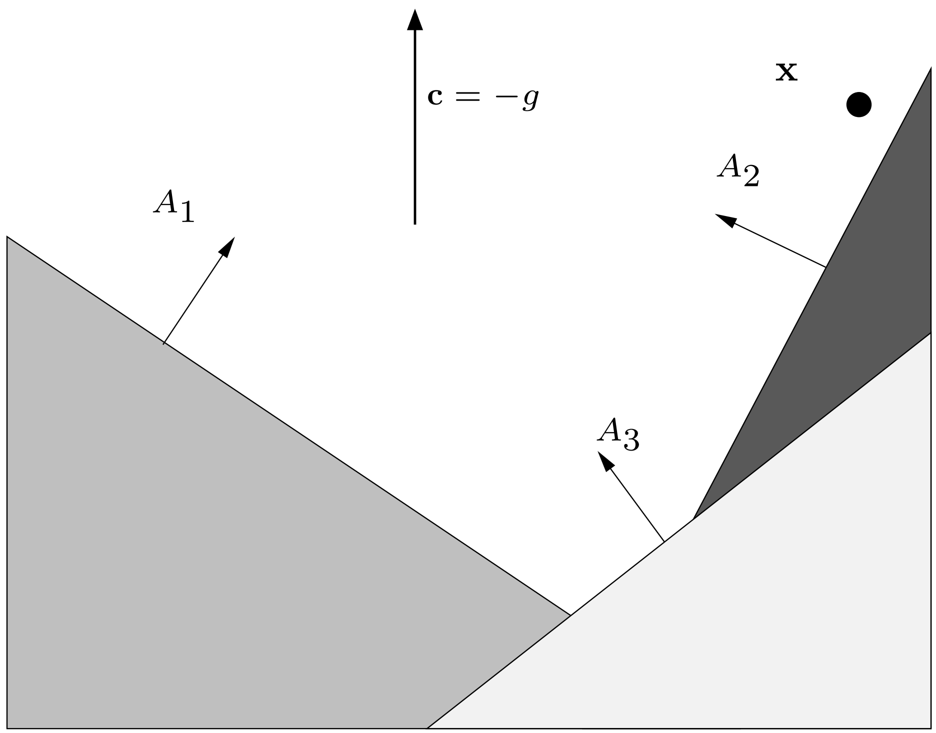
- We have proved that **Farkas' lemma**, a consequence of the isolation theorem, can prove **strong duality**.
- We follow with a widely used geometric and physical **illustration of strong duality**.
- Suppose we are in $\mathbf{R}^2$. We define a set of $m$ inequalities $A_i^T \mathbf{x} \geq b_i$.
- A **ball** is thrown in the feasible set. **Gravity** makes it roll down to the lowest corner of the polyhedron.
- When in contact with the ball, each wall $i$ exerts a force $\mu_i A_i$ on the ball that is parallel to $A_i$.

# Gravity Example

- the position $\mathbf{x}$ of the ball is the solution of

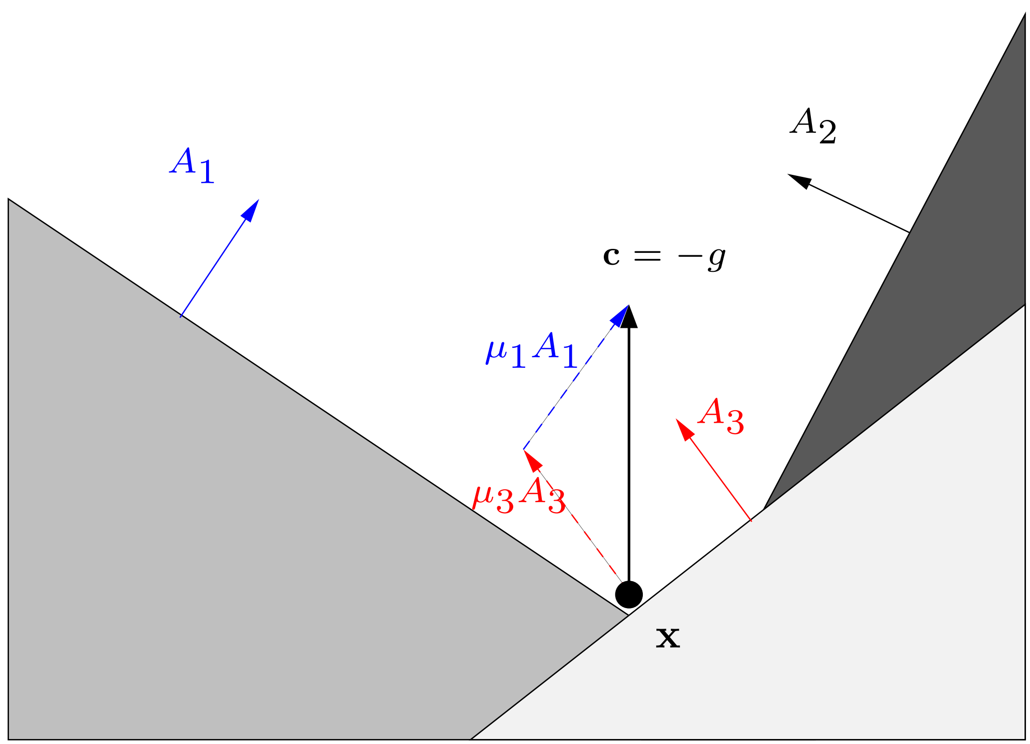
$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A_i^T\mathbf{x} \geq b_i,\ i = 1..m \end{array},$$

where $\mathbf{c}$ points upwards, that is the opposite of the gravity vector.



# Gravity Example

- The different walls exert forces $\mu_1 A_1, \mu_2 A_2, \cdots, \mu_m A_m$ on the ball. $\mu_i \geq 0$
- When $\mathbf{x}$ does not rest on wall $i$, $\mu_i$=0 necessarily. Hence $\mu_i(b_i - A_i^T \mathbf{x}) = 0$.
- At the optimum, the forces cancel gravity: $\sum_{i=1} \mu_i A_i = \mathbf{c}$.
- At the optimum, $\mu^T \mathbf{b} = \sum_{i=1}^m \mu_i b_i = \sum_{i=1}^m \mu_i A_i^T \mathbf{x}^\star = \mathbf{c}^T \mathbf{x} \Rightarrow$ **Strong duality**



# Dual Simplex Method

# Intuition

- Strong duality proof using the simplex:
  - We start with a **standard form minimization**.
  - The **reduced cost coefficient** at the optimum satisfies $c^T - c_{\mathbf{I}}^T B_{\mathbf{I}}^{-1} A \geq 0$.
  - We saw that writing $\mu^T = c_{\mathbf{I}}^T B_{\mathbf{I}}^{-1}$ yielded a **feasible dual solution**.
  - That dual solution was **optimal** and shared the same objective with $\mathbf{x_I}$.

- There is some **obvious symmetry** between the **reduced cost coefficient** and the **solution** for a given base.

# Intuition

- Primal and dual simplex in a few words:
  - Given a BFS for the primal, the **primal** simplex looks for a **a dual feasible solution** $\mu^T = c_{\mathbf{I}}^T B_{\mathbf{I}}^{-1}$ while maintaining **primal feasibility** for $\mathbf{x}$.
  - Given a dual BFS, the **dual** simplex looks for a **a primal feasible solution** $\mathbf{x}$ while maintaining **dual feasibility** for $\mu$.
- Why consider it? great for understanding. useful for sensitivity analysis.

# Tableau

- A not so distant reminder on tableaux

| | | | |
|---|---|---|---|
| $\ddots$ | $\cdots$ | $\ddots$ | $\vdots$ |
| $\vdots$ | $B_{\mathbf{I}}^{-1}A$ | $\vdots$ | $B_{\mathbf{I}}^{-1}\mathbf{b}$ |
| $\ddots$ | $\cdots$ | $\ddots$ | $\vdots$ |
| $\cdots$ | $(\mathbf{c}-\mathbf{c}_{\mathbf{I}}^T B_{\mathbf{I}}^{-1}A)^T$ | $\cdots$ | $-\mathbf{c}_{\mathbf{I}}^T B_{\mathbf{I}}^{-1}\mathbf{b}$ |

- In the dual simplex iterations,
  - we do not assume that $B_{\mathbf{I}}^{-1}\mathbf{b}$ is nonnegative at each iteration.
  - we assume that $(\mathbf{c}-\mathbf{c}_{\mathbf{I}}^T B_{\mathbf{I}}^{-1}A)^T \geq 0$, or equivalently that $\mu^T A \leq \mathbf{c}^T$.
- This means $\mu = B_{\mathbf{I}}^{-1}\mathbf{c}_{\mathbf{I}}$ is dual-feasible...
- Note the analogy between $\mathbf{c}_{\mathbf{I}}^T B_{\mathbf{I}}^{-1}$ or $B_{\mathbf{I}}^{-1}\mathbf{c}_{\mathbf{I}}$ and $B_{\mathbf{I}}^{-1}\mathbf{b}$.
- If by any chance both $(\mathbf{c}-\mathbf{c}_{\mathbf{I}}^T B_{\mathbf{I}}^{-1}A)^T \geq 0$ and $B_{\mathbf{I}}^{-1}\mathbf{b} \geq 0$ then we have found the solution.
- If not... basis change!

# Pivot

- Let's write $\mathbf{r} = \mathbf{c} - \mathbf{c}_{\mathbf{I}}^T B_{\mathbf{I}}^{-1} A$.
- Select a **primal** variable $i_l$ s.t. $(B_{\mathbf{I}}^{-1}\mathbf{b})_l < 0$ and consider the tableau $l$th **row**.
- That row is made of $(y_{l,i})_{1 \leq i \leq n}$ coordinates.
  - for each $i$ such that $y_{l,i} < 0$, consider the ratio $\frac{r_i}{|y_{l,i}|}$,
  - let $j$ be the column number for which this ratio is smallest.
  - $j$ must correspond to a nonbasic variable (otherwise $y_{l,j}$ is zero or 1 for $y_{l,i_l}$).
  - Then completely standard pivot on $y_{l,j}$: $\mathbf{I} \leftarrow \mathbf{I} \setminus \{i_l\} \cup \{j\}$.
  - Can prove that the new reduced cost coefficients stay positive, and we keep dual-feasibility.

# Dual Simplex Pivot Example

- The following tableau is dual feasible

| | | | | | |
|---|---|---|---|---|---|
| $-2$ | $4$ | $1$ | $1$ | $0$ | $2$ |
| $4$ | $\mathbf{-2}$ | $-3$ | $0$ | $1$ | $-1$ |
| $2$ | $6$ | $10$ | $0$ | $0$ | $0$ |

- The basis $\mathbf{I} = \{4, 5\}$. The current solutions' second variable $(B_{\mathbf{I}}^{-1}\mathbf{b})_2$ is negative.
- Negative entries for the second row can be found in 2nd and 3rd variables
- Corresponding ratios $6/|-2|$ and $10/|-3|$. Therefore $\mathbf{I}' = \{4, 5\} \setminus \{5\} \cup \{2\}$ and we pivot accordingly

| | | | | | |
|---|---|---|---|---|---|
| $6$ | $0$ | $-5$ | $1$ | $2$ | $\mathbf{0}$ |
| $-2$ | $1$ | $3/2$ | $0$ | $-1/2$ | $\mathbf{1/2}$ |
| $\mathbf{14}$ | $\mathbf{0}$ | $\mathbf{1}$ | $\mathbf{0}$ | $\mathbf{3}$ | $\mathbf{-3}$ |

- primal and dual feasible... optimal and optimum is 3

# Dual Simplex Summary

- The dual simplex proceeds in the same way that the (primal simplex)
- **Any** base $\mathbf{I}$, defines a primal $B_{\mathbf{I}}^{-1}\mathbf{b}$ and dual solution $(\mathbf{c}_{\mathbf{I}}^T B_{\mathbf{I}}^{-1})A \leq \mathbf{c}^T$.
- Assume $\mathbf{I}$ provides a **dual feasible** solution.
- Update the base through two criterions:
  - The column $B_{\mathbf{I}}^{-1}\mathbf{b}$ has negative elements? that gives **exiting index** $i_l$.
  - Is there a pivot feasible for the reduced costs? **entering column** $j$.
- Pivot and update the whole tableau.

# Dual Simplex Summary

- When can/should we use the dual simplex?
  - We have a base $\mathbf{I}$ that is dual-feasible to start our problem.
  - We have a solution $\mathbf{x}^\star$ with base $\mathbf{I}$ for a problem and we only change the constraints $\mathbf{b}$.

# Sensitivity Analysis

# Sensitivity analysis

- Let's study sensitivity with a generic problem and its dual:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array} \qquad \begin{array}{ll} \text{maximize} & g(\lambda, \mu) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

- Consider a small **perturbation** $(\mathbf{u}, \mathbf{v})$ to the constraints:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq \boldsymbol{u_i}, \quad i = 1, \ldots, m \\ & h_i(x) = \boldsymbol{v_i}, \quad i = 1, \ldots, p \end{array} \qquad \begin{array}{ll} \text{maximize} & g(\lambda, \mu) - \lambda^T \mathbf{u} - \mu^T \mathbf{v} \\ \text{subject to} & \lambda \geq 0 \end{array}$$

- Here $x, \ \lambda, \ \mu$ are variables and $(\mathbf{u}, \mathbf{v})$ parameters.
- We write $p^\star(\mathbf{u}, \mathbf{v})$ for the optimum of the problem given perturbations $\mathbf{u}, \mathbf{v}$.
- This value may not be defined is the problem is unfeasible...

# Global sensitivity analysis

- Suppose we have **strong duality** in the original problem, $i.e. \exists \lambda^\star \geq 0, \mu^\star$ s.t. $P^\star(\mathbf{0},\mathbf{0}) = g(\lambda^\star, \mu^\star)$.
- For $(\mathbf{u},\mathbf{v})$ such that $p^\star(\mathbf{u},\mathbf{v})$ is defined, by weak duality,

$$\begin{aligned} p^\star(u,v) &\geq g(\lambda^\star,\nu^\star) - u^T\lambda^\star - v^T\mu^\star \\ &\geq p^\star(0,0) - u^T\lambda^\star - v^T\mu^\star \end{aligned}$$

- This gives a global lower bound, and indications on $p^\star$ for some changes:
  - If $\lambda_i^\star \gg 0$, $u_i < 0$ (tighten constraint), then **big increase** for $p^\star$.
  - If $\lambda_i^\star$ is small, $u_i > 0$ (loosen constraint), then **little impact** on $p^\star$.
  - If $\mu_i^\star \gg 0$ and $v_i < 0$ or $\mu_i^\star \ll 0$ and $v_i > 0$ then **big increase** for $p^\star$.
  - If $\mu_i^\star \approx 0^+$ and $v_i > 0$ or $\mu_i^\star \approx 0^-$ and $v_i < 0$ then **little impact** on $p^\star$.

# Local sensitivity analysis

- Suppose $p^\star$ is differentiable around $\mathbf{u} = \mathbf{0}, \mathbf{v} = \mathbf{0}$.
- Hence, for small values $(u, v)$ we have:

$$\lambda_i^\star = -\frac{\partial p^\star(0,0)}{\partial u_i}, \quad \mu_i^\star = -\frac{\partial p^\star(0,0)}{\partial v_i}$$

- The dual solution gives the local **sensitivities** of the optimal objective with respect to constraint perturbations.
- This time the interpretation is **symmetric**.
- The objective moves by $-\lambda_i^\star u_i$ whatever the signs of $\lambda_i^\star$ and $u_i$.

# Sensitivity Analysis, The LP case

# Sensitivity analysis, the LP case

- Suppose we have a standard form LP

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

- As usual, assume $\mathbf{I}$ is the optimal base and $\mathbf{x}^\star$ the optimum.
- Suppose $\mathbf{b}$ is replaced by $\mathbf{b} + \mathbf{d}$ where $\mathbf{d} \approx 0$.
  - As long as $\mathbf{x}^\star$ is non-degenerate and $\mathbf{d}$ small, $B_{\mathbf{I}}^{-1}(\mathbf{b} + \mathbf{d}) \geq 0$. *feasible*
  - Since $\mathbf{I}$ is optimal, $\mathbf{c} - \mathbf{c}_{\mathbf{I}}^T B_{\mathbf{I}}^{-1} A \geq 0$. *still optimal*
- Hence the **same basis** is still **optimal** for an infinitesimally perturbed problem.

# Sensitivity analysis, the LP case

- The new optimum is

$$\mathbf{c}_{\mathbf{I}}^T B_{\mathbf{I}}^{-1}(\mathbf{b}+\mathbf{d}) = \mu^T(\mathbf{b}+\mathbf{d})$$

- perturbation $\mathbf{d}$: $z^\star$ becomes $z^\star + \mu^T\mathbf{d}$.
- **each component** $\mu_i$ can be interpreted as the **marginal cost** of each unit increase of $b_i$.
- Such marginal costs are also called *shadow prices*.

# Sensitivity: examples

- The **simplex** can handle more **advanced perturbation scenarios**.
- Suppose we have converged to an optimum $\mathbf{I}$ and have access to $\mathbf{x}^\star$ and $\mu^\star$.
- We review the following scenarios:

1. A new variable is added
2. A new inequality constraint is added
3. A new equality constraint is added
4. The constraint vector $\mathbf{b}$ is changed
5. The cost vector $\mathbf{c}$ is changed
6. A nonbasic column of $A$ changes
7. A basic columns of $A$ changes

and discuss how we can still use $\mathbf{I}$ to get the new optimum **quickly**.

# 1. New variable

- Suppose the program becomes

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} + c_{n+1}x_{n+1} \\ \text{subject to} & A\mathbf{x} + \mathbf{a}_{n+1}x_{n+1} = \mathbf{b} \\ & \mathbf{x} \geq 0, x_{n+1} \geq 0 \end{array}$$

- Note that $(\mathbf{x}^\star, 0)$ is **already a BFS** of the new problem.
- For the basis $\mathbf{I}$ to remain optimal, we need that

$$c_{n+1} - \mathbf{c}_\mathbf{I}^T B_\mathbf{I}^{-1}\mathbf{a}_{n+1} \geq 0.$$

- If this is the case, $\mathbf{I}$ is still optimal.
- If not, we start from $(\mathbf{x}^\star, 0)$ and use the simplex algorithm.
- Running time typically much lower than rerunning everything from scratch.

# 2. New inequality

- Suppose the program has a new constraint $A_{m+1}^T \mathbf{x} \geq b_{m+1}$.
- If $\mathbf{x}^\star$ **already satisfies** this inequality, then $\mathbf{x}^\star$ is still optimal.
- if **not**, introduce a surplus variable $x_{n+1}$ and $A_{m+1}^T \mathbf{x} - x_{n+1} = b_{m+1}$.
- We obtain the following standard form, writing $\beta = \begin{bmatrix} \mathbf{b} \\ b_{m+1} \end{bmatrix}$ and $\mathbf{x} \in \mathbf{R}^{n+1}$,

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \begin{bmatrix} A & \mathbf{0} \\ A_{m+1}^T & -1 \end{bmatrix} \mathbf{x} = \beta \\ & \mathbf{x} \geq 0 \end{array}$$

# 2. New inequality

- We use a basis $\mathbf{I}' = \mathbf{I} \cup \{n+1\}$. Write $\mathbf{a} = A_{m+1,\mathbf{I}}$ Note that

$$B_{\mathbf{I}'} = \begin{bmatrix} B_{\mathbf{I}} & \mathbf{0} \\ \mathbf{a}^T & -1 \end{bmatrix}, \qquad \det B_{\mathbf{I}'} = -\det B_{\mathbf{I}} \neq 0, \qquad B_{\mathbf{I}'}^{-1} = \begin{bmatrix} B_{\mathbf{I}}^{-1} & \mathbf{0} \\ \mathbf{a}^T B_{\mathbf{I}}^{-1} & -1 \end{bmatrix}.$$

- The corresponding primal point is $\begin{bmatrix} \mathbf{x}^\star & \mathbf{a}^T\mathbf{x}^\star - b_{m+1} \end{bmatrix}$. It is infeasible by assumption.

- On the other hand the new reduced cost is given by

$$\begin{bmatrix} \mathbf{c}^T & 0 \end{bmatrix} - \begin{bmatrix} \mathbf{c}_{\mathbf{I}}^T & 0 \end{bmatrix} \begin{bmatrix} B_{\mathbf{I}}^{-1} & \mathbf{0} \\ \mathbf{a}^T B_{\mathbf{I}}^{-1} & -1 \end{bmatrix} \begin{bmatrix} A & \mathbf{0} \\ A_{m+1}^T & -1 \end{bmatrix} = \begin{bmatrix} \mathbf{c}^T - \mathbf{c}_{\mathbf{I}}^T B_{\mathbf{I}}^{-1} A & 0 \end{bmatrix}$$

which is thus nonnegative by optimality of $\mathbf{I}$.

- Hence $\mathbf{I}'$ is **dual feasible**... dual simplex with the tableau given by $B_{\mathbf{I}'}^{-1}$.

# 3. New equality

- New constraint $A_{m+1}^T \mathbf{x} = b_{m+1}$ and suppose $A_{m+1}^T \mathbf{x}^\star > b_{m+1}$
- The dual of the new problem becomes

$$\begin{array}{ll} \text{maximize} & \mu^T \mathbf{b} \\ \text{subject to} & \begin{bmatrix} \mu^T & \mu_{m+1} \end{bmatrix} \begin{bmatrix} A \\ A_{m+1}^T \end{bmatrix} \leq \mathbf{c}^T. \end{array}$$

  where $\mu_{m+1}$ is a new dual variable associated with the latest constraint.

- If $\mu^\star$ is the optimal dual solution for $\mathbf{I}^\star$, $(\mu^\star, 0)$ is feasible, but we have no base $\mathbf{I}$ that corresponds to $(\mu^\star, 0)$...
- Back to the the primal. We modify it by an *auxiliary problem* with $M \gg 0$

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T \mathbf{x} + M x_{n+1} \\ \text{subject to} & A\mathbf{x} = \mathbf{b} \\ & A_{m+1}^T \mathbf{x} - x_{n+1} = b_{m+1} \\ & \mathbf{x}, x_{m+1} \geq 0 \end{array}$$

- We can then use the approach in (2) by considering $B_{\mathbf{I}'} = \begin{bmatrix} B_{\mathbf{I}} & \mathbf{0} \\ \mathbf{a}^T & -1 \end{bmatrix}$.

# 4. Change in constraint vector $\mathbf{b}$

- Suppose $b_j$ of $\mathbf{b}$ is changed to $b_j + \delta$, that is $\mathbf{b}$ is changed to $\mathbf{b} + \delta \mathbf{e}_j$.
- For what range of $\delta$ will $\mathbf{I}$ remain feasible ? remember that optimality is not affected..
- Let $B_{\mathbf{I}}^{-1} = [\beta_{i,j}]$. The condition $B_{\mathbf{I}}^{-1}(\mathbf{b} + \delta \mathbf{e}_j) \geq 0$ is equivalent to

$$\max_{\{i|\beta_{ij}>0\}} -\frac{(B_{\mathbf{I}}^{-1}\mathbf{b})_i}{\beta_{ij}} \leq \delta \leq \min_{\{i|\beta_{ij}<0\}} -\frac{(B_{\mathbf{I}}^{-1}\mathbf{b})_i}{\beta_{ij}}$$

- For this range, the optimal cost is given by $\mathbf{c}_{\mathbf{I}}^T B_{\mathbf{I}}^{-1}(\mathbf{b} + \delta \mathbf{e}_j) = \mu^{\star T}\mathbf{b} + \delta \mu_j^{\star}$.
- Outside the range, run the dual simplex starting with $\mu^{\star}$.

# 5. Change in cost vector $\mathbf{c}$

- Suppose $c_j$ of $\mathbf{c}$ is changed to $c_j + \delta$, that is $\mathbf{c}$ is changed to $\mathbf{c} + \delta \mathbf{e}_j$.
- Primal feasibility of $\mathbf{I}$ is not affected. However, we need to check $\mathbf{c}_\mathbf{I}^T B_\mathbf{I}^{-1} A \leq \mathbf{c}^T$.
- If $j$ corresponds to a **nonbasic** variable, $\mathbf{c}_\mathbf{I}$ does not change, but we need that

$$-(c_j - \mathbf{c}_\mathbf{I}^T B_\mathbf{I}^{-1} \mathbf{a}_j) \leq \delta.$$

  if this is not ensured, we have to apply a few primal simplex iterations.

- If $j$ corresponds to a **basic** variable, i.e. $i_l = j$, then the condition becomes

$$(\mathbf{c}_\mathbf{I} + \delta \mathbf{e}_l)^T B_\mathbf{I}^{-1} \mathbf{a}_i \leq c_i.$$

- Equivalently, $\delta y_{l,i} \leq c_i - \mathbf{c}_\mathbf{I}^T B_\mathbf{I}^{-1} \mathbf{a}_i$ ensures the solution remains optimal.

# 6. Change in nonbasic column of $A$

- The $i$th coordinate of a **nonbasic** column vector $\mathbf{a}_j$ is changed to $a_{ij} + \delta$.
- If the variable is nonbasic, primal feasibility is not affected.
- Dual feasibility: $c_j - \mu^T(\mathbf{a}_j + \delta \mathbf{e}_i) \geq 0$.
- If this inequality is violated, $j$ can be inserted in the basis, requiring a primal simplex step.

# 7. Change in nonbasic column of $A$

- The $i$th coordinate of a **basic** column vector $\mathbf{a}_j$ is changed to $a_{ij} + \delta$, both feasibility and optimality conditions are affected.
- exercise...

# Next time

- Ellipsoid Method and Polynomial Complexity of the Simplex.