ORF 522

Linear Programming and Convex Analysis

Financial Applications

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Today

- Some applications of LP in finance.
- Portfolio management. Similar to Mean-Variance optimization / Markowitz theory.
- LP duality and the existence of a risk-neutral probability.

An Example from Portfolio Optimization

Simple Portfolio Theory

- n traded financial assets.
- For each asset a (random) return R_j at horizon T. $R = \frac{P_T}{p_0} - 1$.
- R_j is a $[-1, \infty)$-valued random variable. not much more...

Simple Portfolio Theory

- A (long) **portfolio** is a **vector** of $\mathbf{R}^n$ which represents the proportion of wealth invested in each asset.
- Namely $\mathbf{x}$ such that $x_1, \cdots, x_n \geq 0$ and $\sum_{x_i} = 1$.
- In \$ terms, Given M dollars, hold $M \cdot x_i$ of asset i.
- The performance of the portfolio is a random variable, $\rho(\mathbf{x}) = \sum_{i=1}^n x_i R_i$.

- Suppose $\mathbf{x} = \left[\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\right]^T$ in the previous example.
- the realized value for $\rho(\mathbf{x})$ is $\frac{4.1\%}{3} + \frac{5.8\%}{3} + \frac{4.2\%}{3} = 4.7\% = 0.047$.

Simple Portfolio Theory

- For a second, imagine we **know** the actual return **realizations** r_j.
- Where would you invest?

- A bit ambitious.. we're not likely to be able see the future.
- Imagine we can **guess** realistically the expected returns $E(R_j)$.
- For instance, $\mathbb{E}[R_{\text{goog}}] = .5 = 50\%, \mathbb{E}[R_{\text{ibm}}] = .05 = 5\%, \mathbb{E}[R_{\text{dow}}] = .01 = 1\%$.
- If your goal is to maximize expected return,

$$\mathbf{x} = \text{argmax}(\mathbb{E}(\rho(\mathbf{x})),$$

  where would you put your money?

- The other question... **is that really what you want** in the first place?

Risk?

- **PHARMA** is a pharmaceutical company working on a new drug.
  - its researchers (or you) think there is a 50% probability that the new drug works
  - Let's do a binary scenario to keep things simple.
    - **the drug works and is approved by FDA**: PHARMA's market value is multiplied by 3. $R = 2$
    - **the drug does not work**: PHARMA goes bankrupt $R = -1$.
  - **Expected return**: $\mathbb{E}[R_{\text{PHARMA}}] = \frac{2+-1}{2} = 1 = 100\%$. You are *expecting* to double your bet.

- **BORING** is a company that produces and sells screwdrivers.
  - The return is uniformly distributed between $-.01 = -1\%$ and $.02 = 2\%$
  - **Expected return** is $.0005$, that is 0.5%.

- Would you bet everything on PHARMA with these cards? **something is missing in our formulation**

Risk?

- Portfolio optimization needs to input the investor's aversion to risk.
- Using $\mathbf{x} = \operatorname{argmax}(\mathbb{E}(\rho(\mathbf{x}))$ can lead the investor to forget about risk.
- Solution: include risk in the program. Risk is vaguely a **quantification** of the **dispersion** of the returns of a portfolio.
- Different choices:
  - **Variance**:
    - ▷ C is the covariance matrix of the vector r.v. R takes values in $\mathbf{R}^n$, $C = \mathbb{E}[(R - \mathbb{E}[R])(R - \mathbb{E}[R])^T]$.
    - ▷ The variance of $\rho(\mathbf{x})$ is simply $\mathbf{x}^T C\mathbf{x}$.
    - ▷ Maximal expected return under variance constraints = mean-variance optimization.
  - **Mean-absolute deviation (MAD)**:
    - ▷ Namely $\mathbb{E}\left[|(\rho(\mathbf{x}) - \mathbb{E}[\rho(\mathbf{x})])|\right] = E[|\mathbf{x}^T \bar{R}|]$ where $\bar{R} = R - \mathbb{E}[R]$.
    - ▷ Penalized estimation: $\mathbf{x} = \underset{\mathbf{x}\geq 0, \mathbf{x}^T\mathbf{1}_n=1}{\operatorname{argmax}} \underbrace{\lambda}_{\text{trade-off}} \cdot \underbrace{\mathbb{E}[\rho(\mathbf{x})]}_{\text{expected return}} - \underbrace{\mathbb{E}[|\mathbf{x}^T\bar{R}|]}_{\text{risk}}$.

Risk

- The **variance** formulation leads to a quadratic program:

$$\begin{array}{ll} \text{maximize} & \mathbf{x}^T \mathbb{E}[R] \\ \text{subject to} & \mathbf{x} \geq 0, \mathbf{x}^T \mathbf{1}_n = 1 \\ & \mathbf{x}^T C \mathbf{x} \leq \lambda \end{array}$$

- The **MAD** formulation leads to something closer to linear programming:

$$\begin{array}{ll} \text{maximize} & \lambda \mathbf{x}^T \mathbb{E}[R] - \mathbb{E}[|\mathbf{x}^T \bar{R}|] \\ \text{subject to} & \mathbf{x} \geq 0, \mathbf{x}^T \mathbf{1}_n = 1 \end{array}$$

- **Problem**: lots of expectations $\mathbb{E}$...
- We need to fill in some expected values above by some **guesses**.

Approximations

- We write $\tilde{\mathbf{r}}$ for $\mathbb{E}[R]$ which can be guessed according to...
  - research, analysts playing with excel, valuation models.
  - historical returns.
- We also need to approximate $\mathbb{E}[|\mathbf{x}^T\bar{R}|]$.
- Suppose we have a history of N returns $(\mathbf{r}^1, \cdots, \mathbf{r}^N)$ where each $\mathbf{r} \in \mathbf{R}^n$.
  - Write $\bar{\mathbf{r}} = \sum_{j=1}^N \mathbf{r}^j$.
  - in practice, approximate $\mathbb{E}[|\mathbf{x}^T\bar{R}| \approx \sum_{j=1}^N |\mathbf{x}^T(\mathbf{r}^j - \bar{\mathbf{r}})|$
- this becomes:

$$\begin{array}{ll} \text{maximize} & \lambda\mathbf{x}^T\mathbf{r} - \frac{1}{N}\sum_{j=1}^N |\mathbf{x}^T(\mathbf{r}^j - \bar{\mathbf{r}})| \\ \text{subject to} & \mathbf{x} \geq 0, \mathbf{x}^T\mathbf{1}_n = 1 \end{array}$$

- Now add artificial variables $y_j = |\mathbf{x}^T(\mathbf{r}^j - \bar{\mathbf{r}})|$. One for each observation. Now,

$$\begin{array}{ll} \text{maximize} & \lambda\mathbf{x}^T\mathbf{r} - \frac{1}{N}\sum_{j=1}^N y_j \\ \text{subject to} & \mathbf{x} \geq 0, y_j \geq 0, \mathbf{x}^T\mathbf{1}_n = 1, \\ & -y_j \leq \mathbf{x}^T(\mathbf{r}^j - \bar{\mathbf{r}}) \leq y_j, j = 1, \cdots, N \end{array}$$

LP's, Duality and Arbitrage

Duality and Arbitrage

- We propose in this an economic interpretation of duality
- Due to Arrow, Debreu, in the 50's. . .
- Used **every day** on financial markets (sometimes unknowingly)
- Simple LP duality result, but underpins most of modern finance theory. . .

One period model

- As in the previous section, basic discrete, **one period** model on a single asset.
- Its price **today** is q_1. Its (random) price **time T ahead** is x.
- Assume x can only take any of the following values

$$x \in \{x_1, \ldots, x_n\}$$

at a **maturity date** T, and that we have an estimate of their probabilities,

$$\{p_1, \cdots, p_n\}.$$

- We have **discretized** the space of possibilities.
- We can only trade **today** and at **maturity**
- There is a **cash** security worth $\$1$ today, that pays $\$1$ at maturity
- near-zero interest rates. sounds familiar?

One period model

- There are also $m-1$ other securities with payoffs at maturity given by

$$h_k(x_i) \quad \text{if } x = x_i \text{ at time } T$$

  for $k = 2, \ldots, m-1$.

- The payoffs are **arbitrary** functions of the n possible values of the asset at time T.
- We could have $h_k(x) = x^2$. Or that for $i \leq j$, $h_k(x_i) = 0$, $i > j$, $h_k(x_i) = 1$.
- We denote by q_k the price **today** of security k with payoff $h_k(x)$.

All these securities are tradeable, can we use them to get information on the price of **another security** with payoff $h_0(x)$?

Static Arbitrage

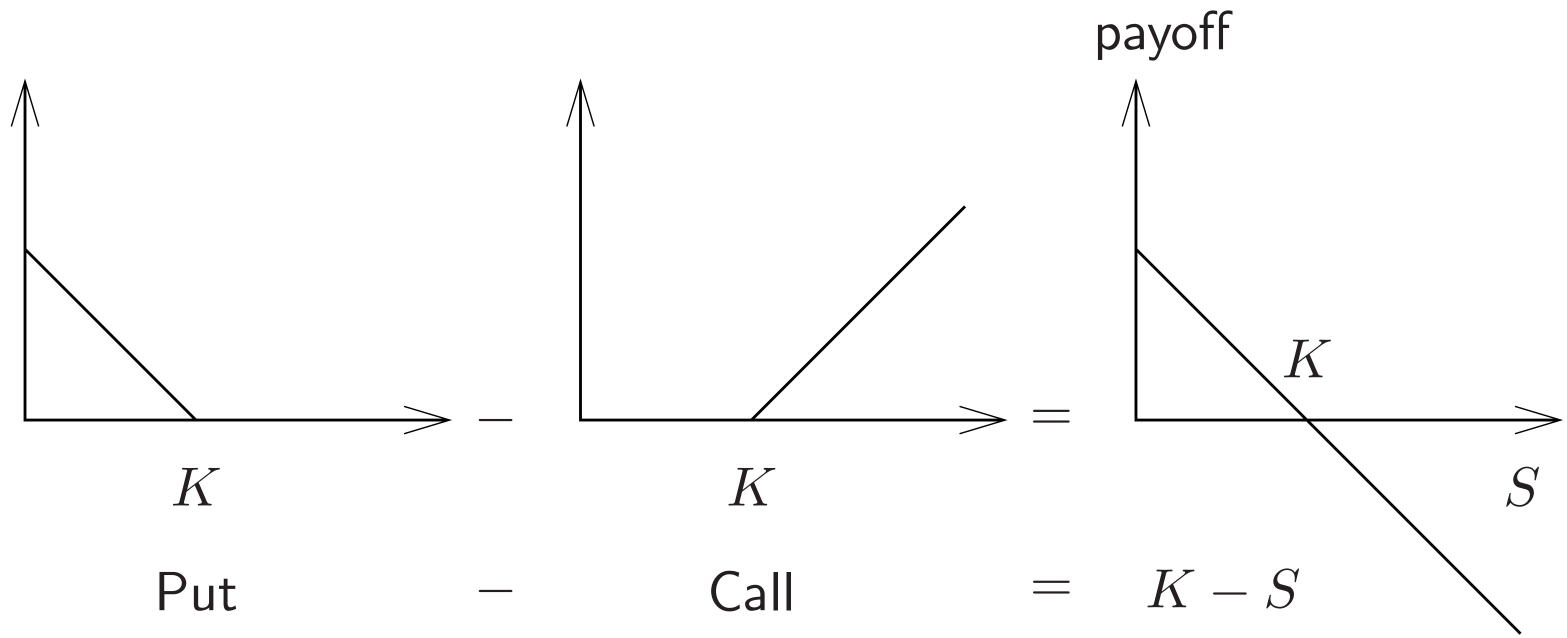
Remember:

- We can only trade today and at maturity.
- We can only trade in securities which are priced by the market.

We want to exclude **arbitrage strategies**

- If the payoff of a portfolio A is always larger than that of a portfolio B then Price(A) $\geq$ Price(B).
- The price of the sum of two products is equal to the sum of the prices.

Simplest Example: Put Call Parity



Put − Call = $K - S$

Price bounds

Suppose that we form a portfolio of cash, stocks and securities $h_k(x)$ with coefficients λ_k:

λ_0	in cash
λ_1	in stock
λ_k	in security $h_k(x)$

- All portfolios that satisfy

$$\lambda_0 + \lambda_1 x_i + \sum_{k=2}^{m} \lambda_k h_k(x_i) \geq h_0(x_i) \quad \text{i=1,. . . ,n}$$

  must be **more expensive** than the security $h_0(x)$
- All portfolios that satisfy the **opposite** inequality must be **cheaper**
- For portfolios that satisfy neither of these, **nothing** can be said. . .
- We are just comparing portfolios dominated for **all** outcomes of x.

Price bounds

- For each of these portfolios, we get an upper/lower bound on the price today of the security $h_0(x)$.
- We can look for optimal bounds. . .
- We can solve:

$$\begin{array}{ll} \text{minimize} & \lambda_0 + \lambda_1 q_1 + \sum_{k=1}^{m} \lambda_k q_k \\ \text{subject to} & \lambda_0 + \lambda_1 x_i + \sum_{k=2}^{m} \lambda_k h_k(x_i) \geq h_0(x_i), \quad i = 1, \ldots, n \end{array}$$

  - Linear program in the variable $\lambda \in \mathbf{R}^{(m+1)}$
  - Produces an optimal upper bound on the price today of the security $h_0(x)$

Linear Programming Duality

- The original linear program looks like:

$$\begin{array}{ll} \text{minimize} & c^T\lambda \\ \text{subject to} & A\lambda \geq b \end{array}$$

  which is a linear program in the variable $\lambda \in \mathbf{R}^m$.

- We can form the Lagrangian

$$L(\lambda, p) = c^T\lambda + y^T(b - A\lambda)$$

  in the variables $\lambda \in \mathbf{R}^m$ and $y \in \mathbf{R}^n$, with $y \succeq 0$.

Linear Programming Duality

- We then minimize in λ to get the dual function

$$g(y) = \inf_{\lambda} c^T\lambda + y^T(b - A\lambda)$$

for $y \succeq 0$, which is again

$$g(y) = \inf_{\lambda} y^T b + \lambda^T(c - A^T y)$$

and we get:

$$g(y) = \begin{cases} y^T b & \text{if } c - A^T y = 0 \\ -\infty & \text{if not.} \end{cases}$$

Linear Programming Duality

- With

$$g(y) = \begin{cases} y^T b & \text{if } c - A^T y = 0 \\ -\infty & \text{if not.} \end{cases}$$

- we get the **dual linear program** as:

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y = c \\ & y \geq 0 \end{array}$$

which is also a linear program in $x \in \mathbf{R}^n$.

LP duality: summary

- The primal LP is the original linear program looks like:

$$\begin{array}{ll} \text{minimize} & c^T\lambda \\ \text{subject to} & A\lambda \geq b \end{array}$$

- its **dual** is then given by:

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y = c \\ & y \geq 0 \end{array}$$

Strong duality: both optimal values are **equal**

LP duality & arbitrage

- Let's look at what this produces for the portfolio problem. . .
  - The **primal** problem in the variable $\lambda \in \mathbf{R}^m$ is given by:

$$p^{\max} := \quad \text{min.} \quad \lambda_0 + \lambda_1 q_1 + \sum_{k=2}^{m} \lambda_k q_k$$

$$\text{s.t.} \quad \lambda_0 + \lambda_1 x_i + \sum_{k=2}^{m} \lambda_k h_k(x_i) \geq h_0(x_i), \quad i = 1, \ldots, n$$

  - The **dual** in the variable $y \in \mathbf{R}^n$ is then

$$p^{\max} := \quad \text{max.} \quad \sum_{i=1}^{n} y_i h_0(x_i)$$

$$\begin{aligned} \text{s.t.} \quad & \sum_{i=1}^{n} y_i h_k(x_i) = q_k, \quad k = 2, \ldots, m \\ & \sum_{i=1}^{n} y_i x_i = q_1 \\ & \sum_{i=1}^{n} y_i = 1 \\ & y \geq 0 \end{aligned}$$

LP duality & arbitrage

- The last two constraints $\{\sum_{i=1}^n y_i = 1,\ y \geq 0\}$ mean that y is a **probability measure**.
- We can rewrite the previous program as:

$$
\begin{array}{rll}
p^{\max} := & \text{max.} & \mathbf{E}_y[h_0(x)] \\
\\
& \text{s.t.} & \mathbf{E}_y[h_k(x)] = q_k, \quad k = 2, \ldots, m \\
& & \mathbf{E}_y[x] = q_1 \\
& & y \text{ is a probability}
\end{array}
$$

- We can compute $p^{\min}$ by minimizing instead.

LP duality & arbitrage

- What does this mean?
- There are three ranges of prices for the security with payoff $h_0(x)$:
  - Prices above $p^{\max}$: these are **not viable**, you can get a cheaper portfolio with a payoff that always dominates $h_0(x)$.
  - Prices in $[p^{\min}, p^{\max}]$: prices are **viable**, $i.e.$ compatible with the absence of arbitrage.
  - Prices below $p^{\min}$: these are **not viable**, you can get a portfolio that is more expensive than $h_0(x)$ with a payoff that is always dominated by $h_0(x)$.

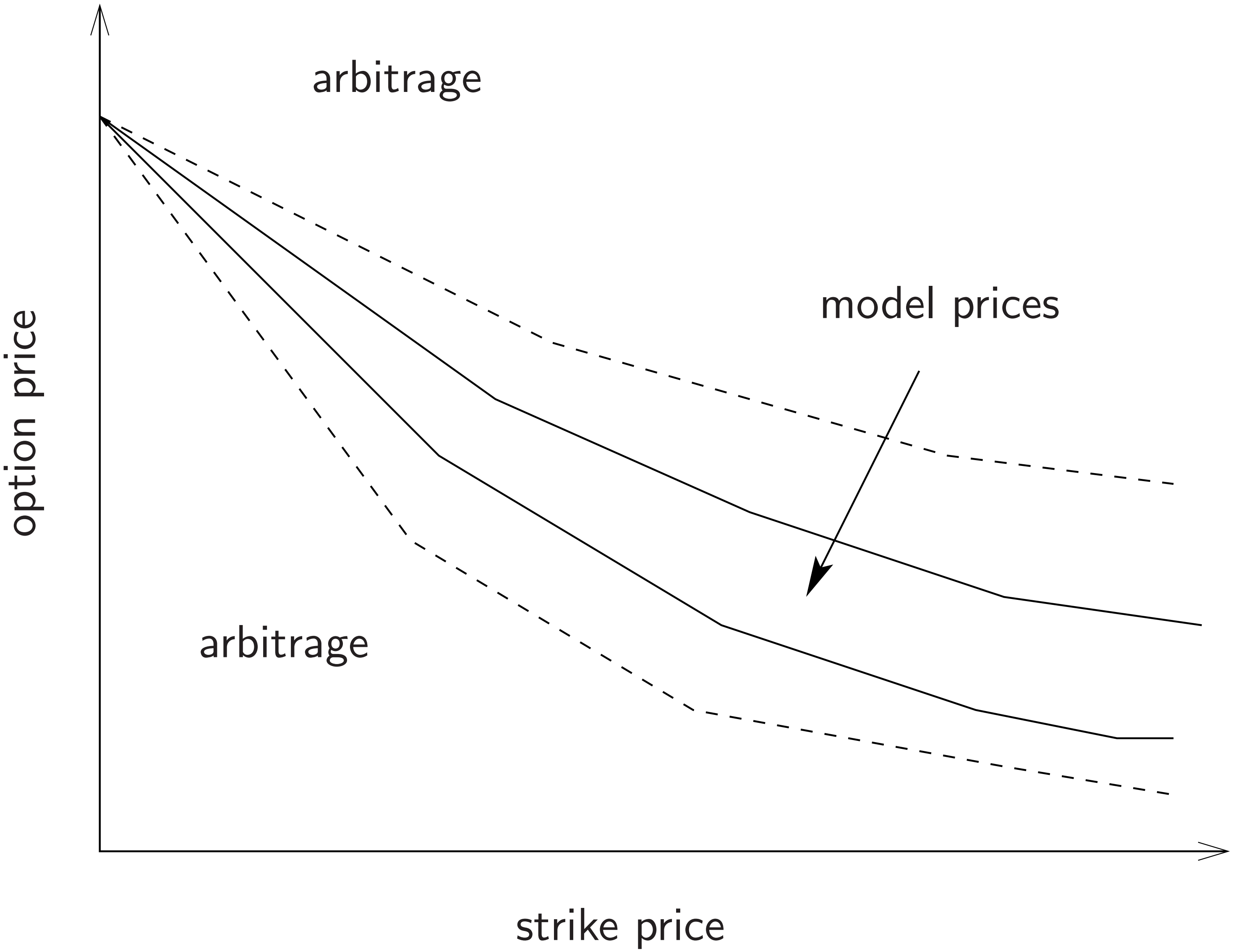
Price bounds

- Example:
  - Suppose the product in the objective is a call option:

$$h_0(x) = (x - K)^+$$

    where K is called the strike price.
  - Suppose also that we know the prices of some other instruments
  - We get upper and lower price bounds on the price of this call for each strike K
- On a graphic. . .

Price Bounds



LP duality & arbitrage

- What if there is no solution y and the linear program is infeasible?
  - Then the original data set q must contain an arbitrage.
  - Start with one product, stock and cash. . . and test.
  - Increase the number of products. . .

LP duality & arbitrage

Fundamental theorem of asset pricing

Theorem 1. *In the one period model, there is no arbitrage between the prices* $\{q_0, \ldots, q_m\}$ *of securities with payoffs at maturity* $\{h_0(x), \ldots, h_m(x)\}$

$$\Updownarrow$$

There exists a probability y *(with* $\sum_{i=1}^{n} y_i = 1$ *and* $y \geq 0$*) such that*

$$q_k = \mathbf{E}_y[h_k(x)], \quad k = 0, \ldots, m$$

LP duality & arbitrage

- Because prices are computed using **expectations under** y (and not expected utility/certain equivalent), we call the probability y **risk-neutral**.
- In particular, it satisfies $q_1 = \mathbf{E}_y[x]$
- If there are *constant* interest rates, simply use **discounted** values for **prices at maturity**. . .
- This probability y has **nothing to do** with the observed distribution of the asset x or its past distribution! (Very common mistake)

LP duality & arbitrage

- Because one can trade
  - the asset
  - derivative products based on the asset

  to form portfolios to hedge/replicate other products, it is possible to evaluate these products using expected value under an **appropriate choice** of probability.

- Again, the risk-neutral probability y is a **tool** **inferred from market prices**,

- it has nothing to do with the statistical properties of the underlying asset x.

- Linear programming duality is interpreted as a duality between **portfolios on assets** problems and **probabilities** (models)

LP duality & arbitrage

In the previous result:

- Set of possible **probabilistic models** = **probability simplex**:
  $p_i \geq 0, \; \sum_i p_i = 1$
- Expected value, hence price is linear in the probability p_i

$$\mathbb{E}[h(x)] = \sum_i p_i h(x_i)$$

- A price constraint is just a linear equality constraint on the probabilities:

$$\sum_i p_i h(x_i) = b_i$$

- Simple family of distributions.

Moment constraints

Choices for asset pricing formulas that depend on the prices directly:. . .

- Use indicator function as payoff:

$$h(x) = 1_{\{x \geq K\}}$$

  to produce the constraint:

$$\sum_i p_i \, 1_{\{x_i \geq K\}} = P(X \geq K) = b$$

- Also, quadratic variation:

$$h(x) = x^2$$

  Corresponds to:

$$\sum_i p_i \, x_i^2 = \mathbb{E}[x_i^2] = b$$

Moment constraints

Higher order formulations? Variance?

- We can't incorporate a variance swap
- A constraint of the form

$$\textbf{Variance}(x) = q_V$$

  why?

- Becomes $\sum_i p_i x_i^2 - (\sum_i p_i x_i)^2 = q_V \Rightarrow$ quadratic constraints in p_i.
- Would however works if we also fix the expected value:

$$\mathbb{E}[x] = b$$

  Corresponds to a **forward** price (EV of the asset):

$$\sum_i p_i \; x_i = q_F \quad \text{and} \quad \textbf{Variance}(x) = \sum_i p_i \; x_i^2 - q_F^2 = q_V$$

- We came back to a simple **linear constraint**

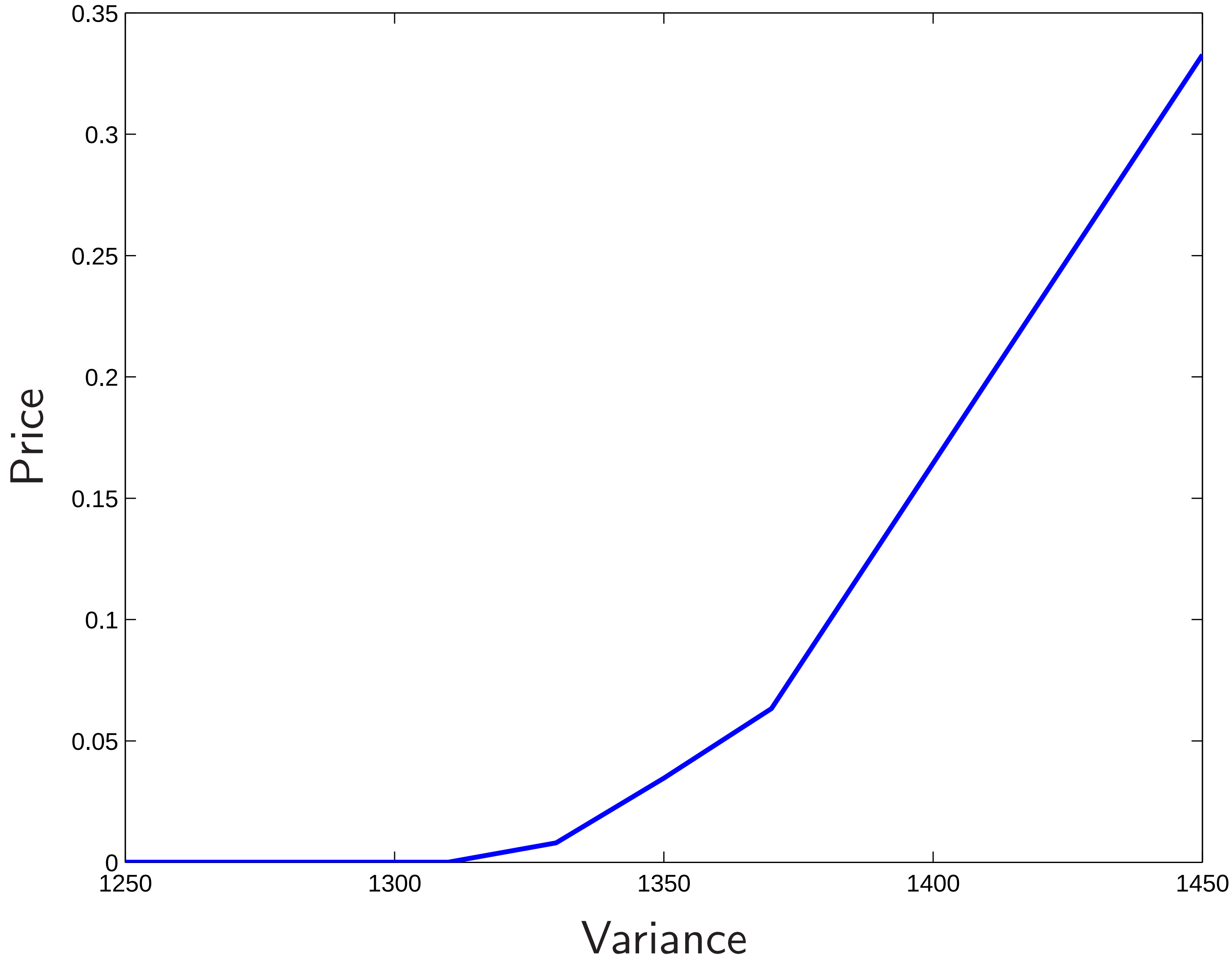
Option price vs. variance

- Fix the forward price (expected value of the asset), **move the variance**. . .
- We study the price of a **call option** h_0.

$$
\begin{aligned}
\text{maximize} \quad & \sum_i p_i \; h_0(x_i) \\
\text{subject to} \quad & \sum_i p_i \; x_i = S_0 \\
& \sum_i p_i \; x_i^2 = b^2 \\
& 0 \le p_i \le 1,
\end{aligned}
$$

- Look at the price as a function of b^2. . .

Option price vs. variance



Option pricing & LP: example

Option pricing

Option pricing example. . .

- Study the price **CutCall** option, with payoff:

$$h_0(X) = (X - K)^+ 1_{\{X \leq L\}}$$

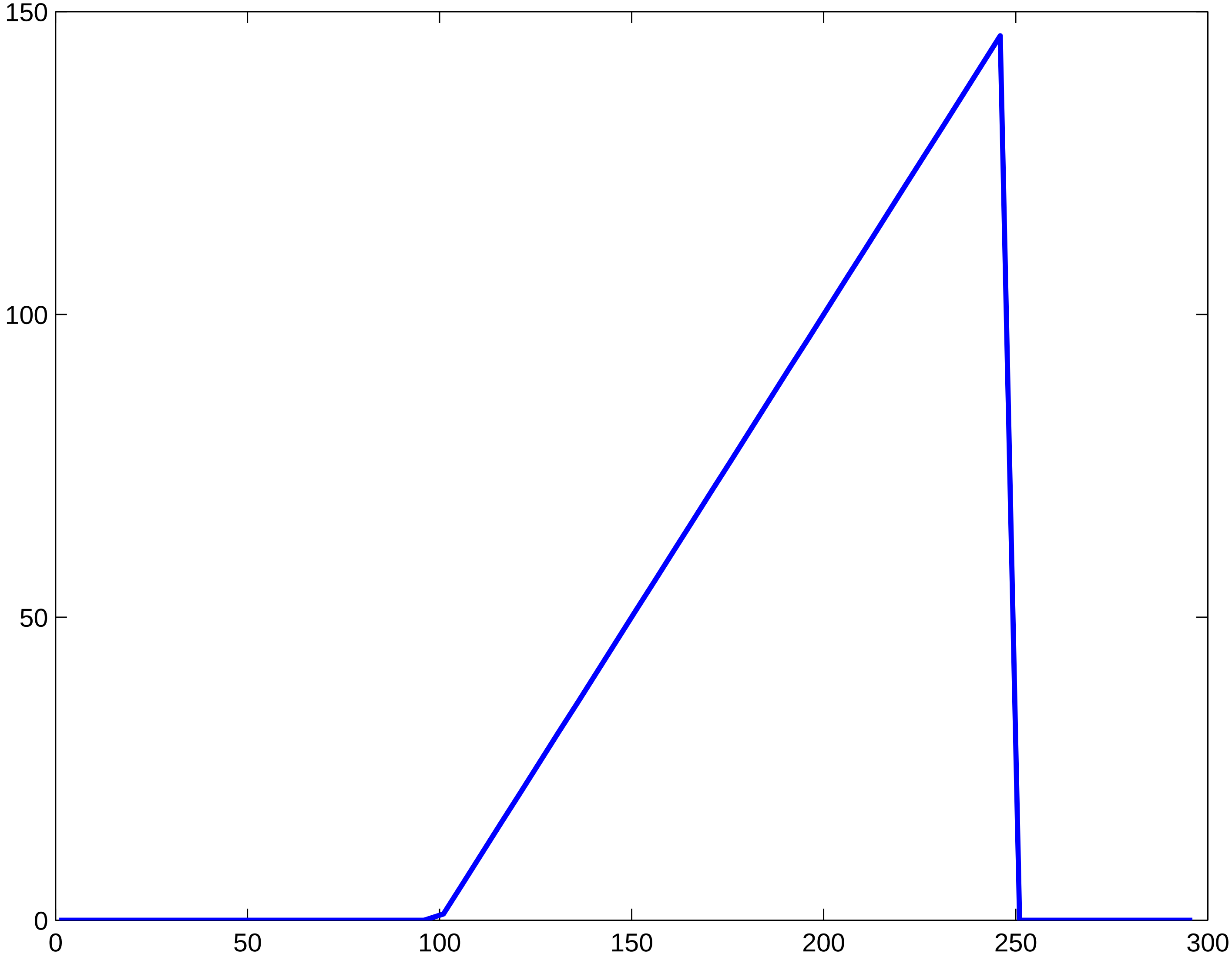
- Similar to knock-out option but only **check at maturity**. **No knock-out** during its life, **european** kind of knock-out.
- Get some market prices q_k for **regular** calls:

$$h_k(X) = (X - K_k)^+$$

- Solve for the maximum CutCall price:

$$\begin{array}{ll} \text{maximize} & \sum_i p_i h_0(x_i) \\ \text{subject to} & \sum_i p_i h_k(x_i) = q_k \\ & \sum_i p_i = 1 \\ & p_i \geq 0 \end{array}$$

Payoff



Option pricing

Solve

$$\begin{array}{ll} \text{maximize} & \sum_i p_i h_0(x_i) \\ \text{subject to} & \sum_i p_i h_k(x_i) = q_k \\ & \sum_i p_i = 1 \\ & p_i \geq 0 \end{array}$$

with

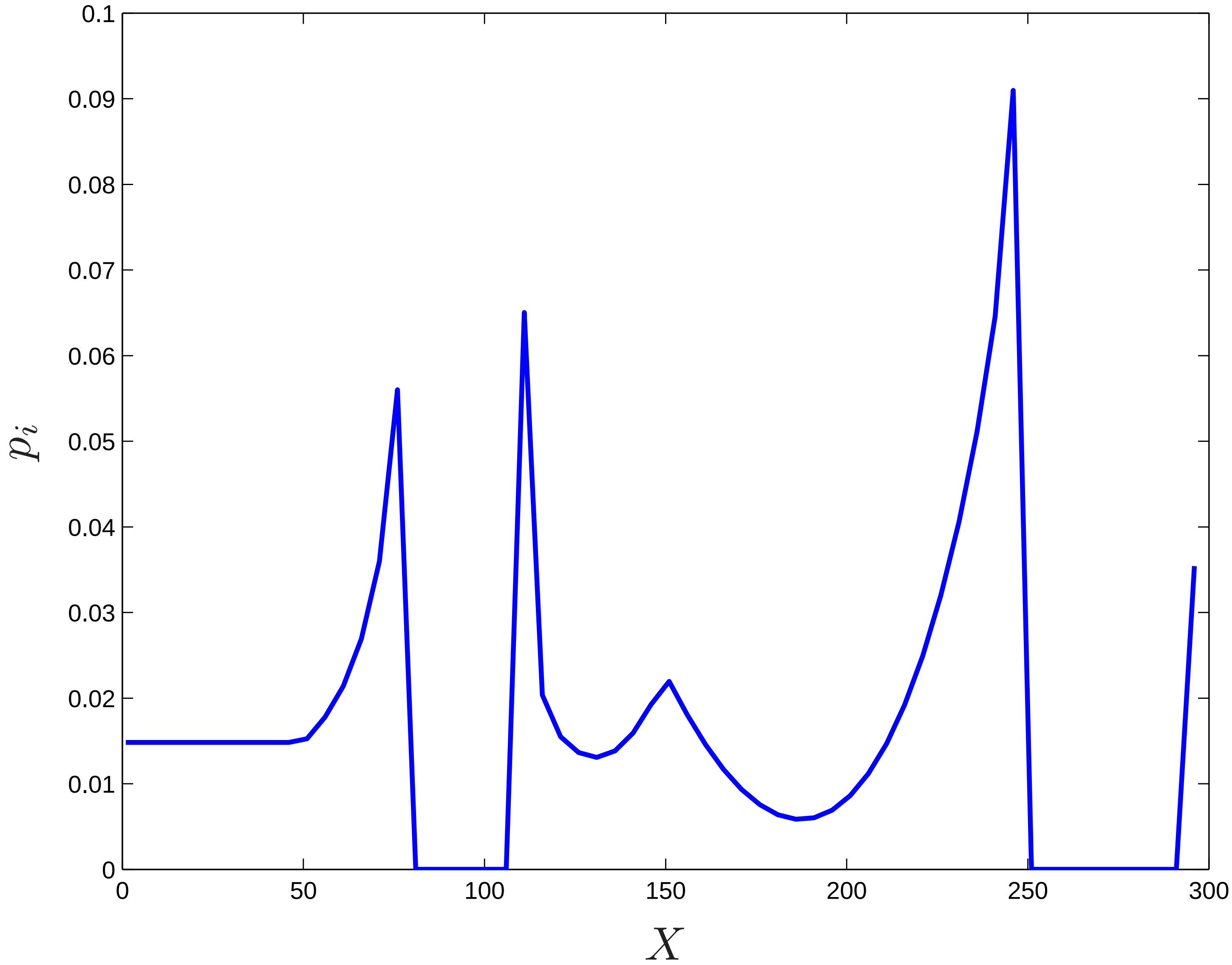
$$K = \{50, 80, 110, 120, 150, 280\}$$

and vector of prices for the 6 options.

$$q = (102.9167, 79.5667, 59.2167, 53.1000, 36.7500, 0.5667)$$

- Prices were computed above using the **uniform** distribution on $[0, 300]$
- **Result**: maximum price for the CutCall is **59**
- Next slide: risk neutral distribution for that maximal price.

Corresponding Risk-Neutral Probability



Option pricing

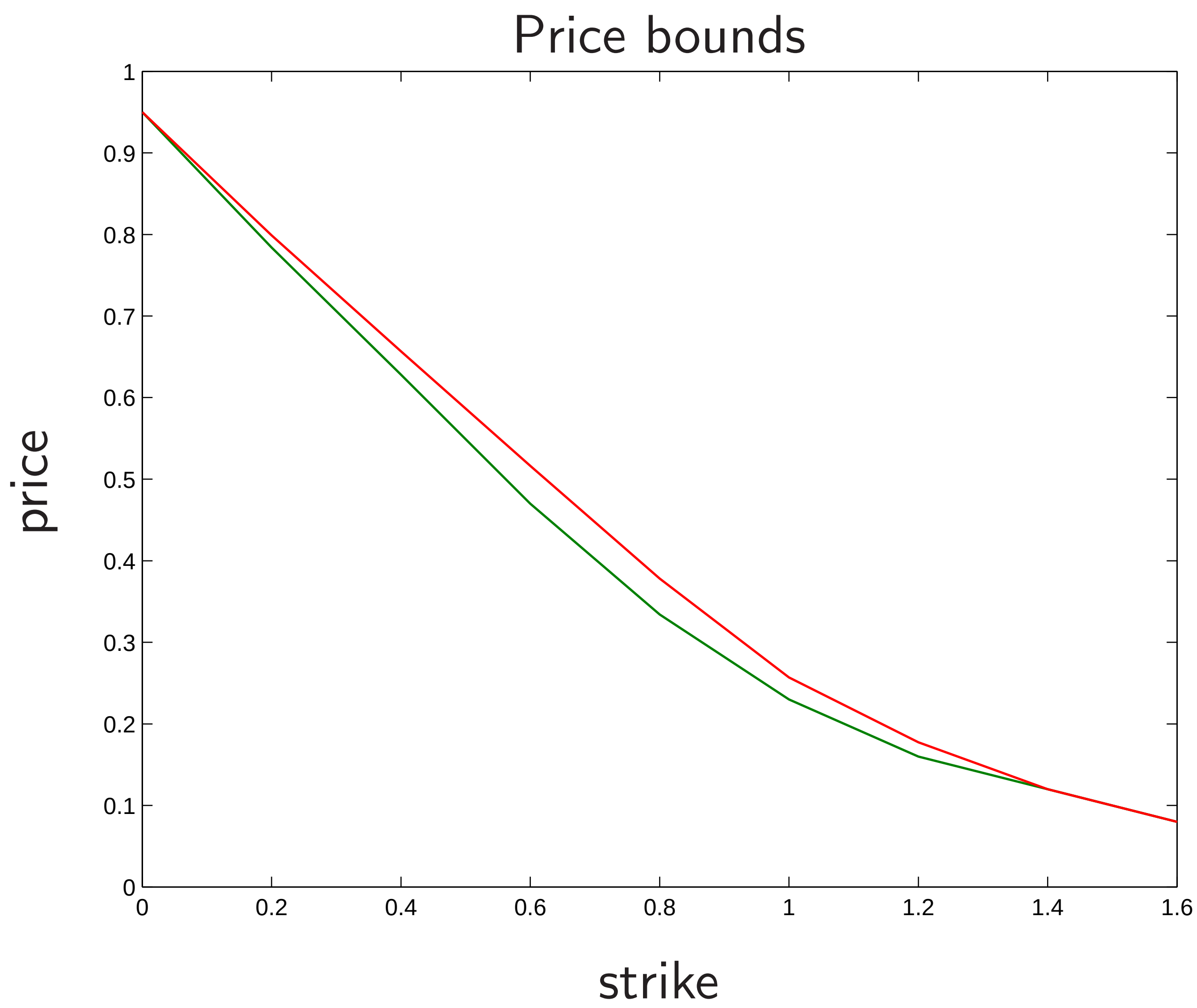
- Problem in dimension 2, price a **basket options** with payoff

$$(x_1 + x_2 - K)_+$$

- The input data set is composed of the asset prices together with the following call prices:

$$(.2x_1 + x_2 - .1)_+,\ (.5x_1 + .8x_2 - .8)_+,$$
$$(.5x_1 + .3x_2 - .4)_+,\ (x_1 + .3x_2 - .5)_+,$$
$$(x_1 + .5x_2 - .5)_+,\ (x_1 + .4x_2 - 1)_+,$$
$$(x_1 + .6x_2 - 1.2)_+.$$

Option pricing



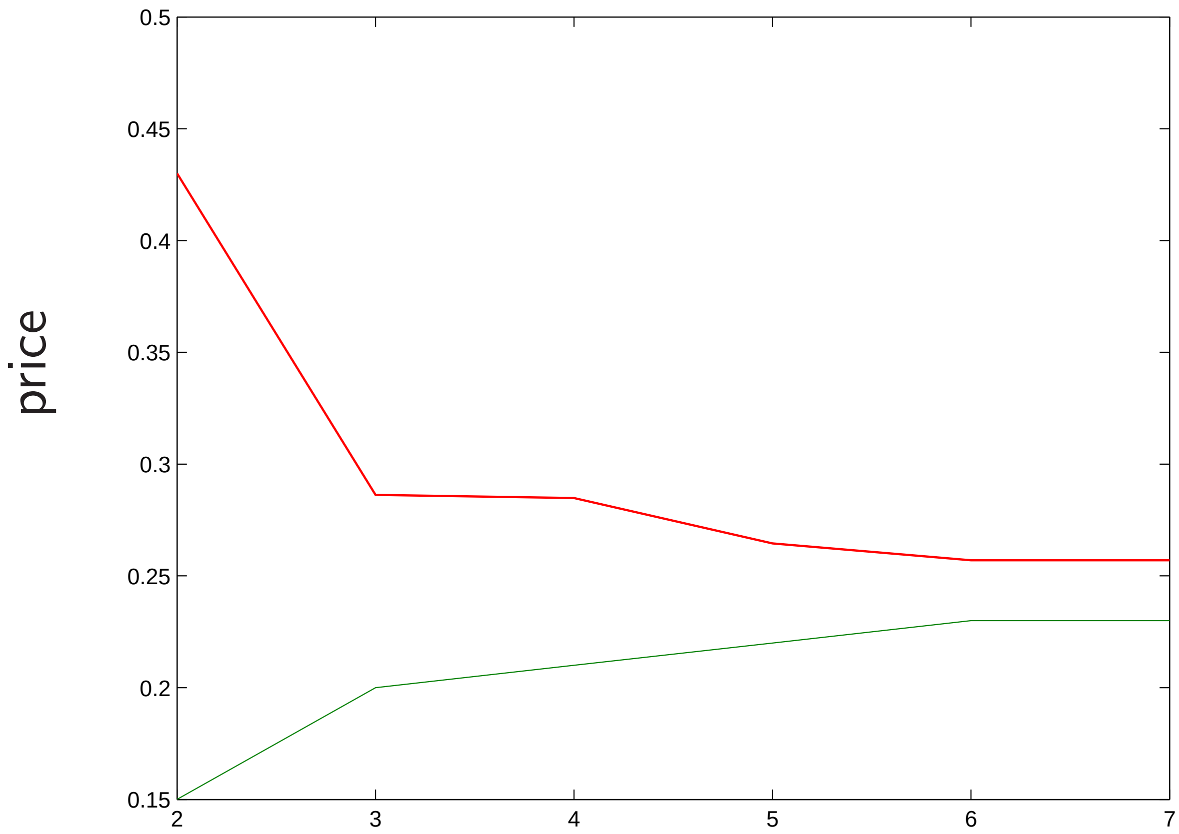
Option pricing

Run another test:

- Look at how these bounds evolve as more and more instruments are incorporated into the data set.
- Fix $K = 1$, we compute the bounds using only the k first instruments in the data set, for $k = 2, \ldots, 7$.
- Plot the **upper** and **lower** bounds
- Also plot one of the solutions

Conclusion: **more market values $\Rightarrow$ tighter bounds**

Option pricing



Option pricing

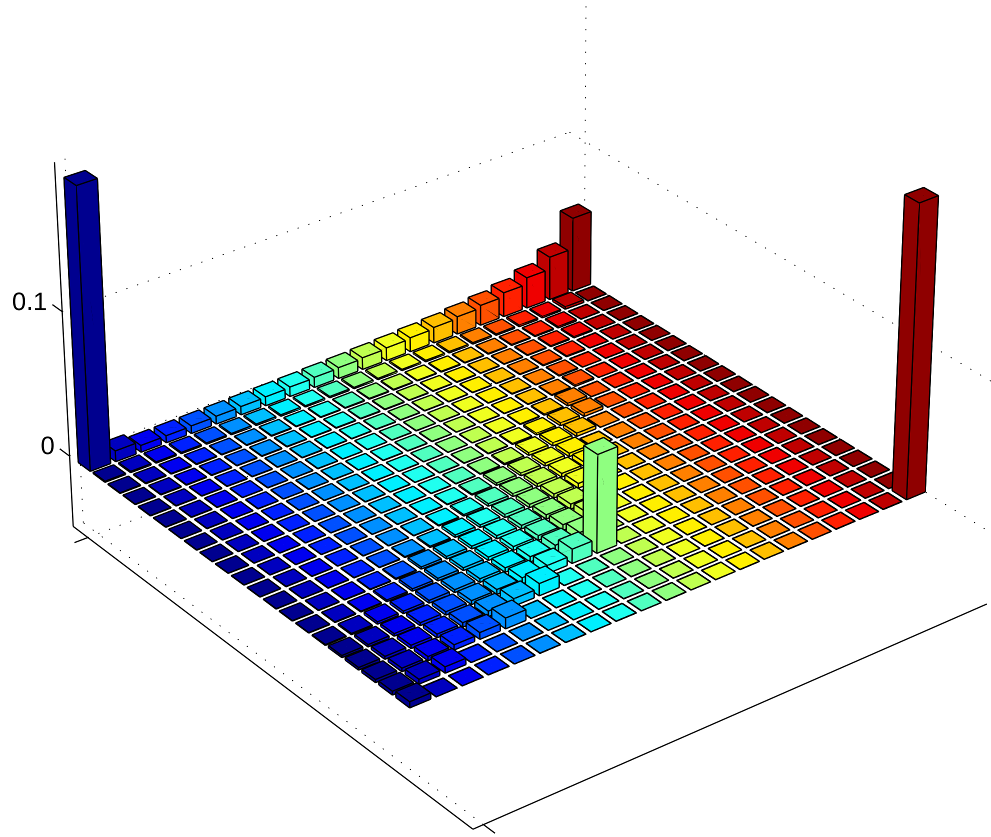


Figure 1: Example of discrete distribution minimizing the price of $(x_1 + x_2 - K)_+$.

Caveats

Size!

- Grows **exponentially** in k^n with the number of points
- Only works with **discrete** and **bounded** models

Everything comes at a price. . .