ORF 522

Linear Programming and Convex Analysis

Financial Applications

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Today

- Some applications of LP in finance.
- Portfolio management. Similar to Mean-Variance optimization / Markowitz theory.
- LP duality and the existence of a risk-neutral probability.
An Example from Portfolio Optimization
Simple Portfolio Theory

- $n$ traded financial assets.
- For each asset $a$ (random) return $R_j$ at horizon $T$. $R = \frac{P_T}{P_0} - 1$.
- $R_j$ is a $[-1, \infty)$-valued random variable. not much more...
A (long) **portfolio** is a vector of $\mathbb{R}^n$ which represents the proportion of wealth invested in each asset.

Namely $x$ such that $x_1, \cdots, x_n \geq 0$ and $\sum x_i = 1$.

In $\$ \text{ terms, Given } M \text{ dollars, hold } M \cdot x_i \text{ of asset } i$.

The performance of the portfolio is a random variable, $\rho(x) = \sum_{i=1}^{n} x_i R_i$.

Suppose $x = [\frac{1}{3} \frac{1}{3} \frac{1}{3}]^T$ in the previous example.

the realized value for $\rho(x)$ is $\frac{4.1\%}{3} + \frac{5.8\%}{3} + \frac{4.2\%}{3} = 4.7\% = 0.047$. 
• For a second, imagine we **know** the actual return realizations $r_j$.

• Where would you invest?

• A bit ambitious.. we’re not likely to be able see the future.

• Imagine we can **guess** realistically the expected returns $E(R_j)$.

• For instance, $E[R_{goog}] = .5 = 50\%, E[R_{ibm}] = .05 = 5\%, E[R_{dow}] = .01 = 1\%$.

• If your goal is to maximize expected return,

$$x = \text{argmax}(E(\rho(x)))$$

where would you put your money?

• The other question... **is that really what you want** in the first place?
Risk?

- **PHARMA** is a pharmaceutical company working on a new drug.
  - its researchers (or you) think there is a 50% probability that the new drug works
  - Let’s do a binary scenario to keep things simple.
    - **the drug works and is approved by FDA**: PHARMA’s market value is multiplied by 3. \( R = 2 \)
    - **the drug does not work**: PHARMA goes bankrupt \( R = -1 \).
  - **Expected return**: \( \mathbb{E}[R_{PHARMA}] = \frac{2 + (-1)}{2} = 1 = 100\% \). You are expecting to double your bet.

- **BORING** is a company that produces and sells screwdrivers.
  - The return is uniformly distributed between \(-0.01 = -1\%\) and \(0.02 = 2\%\)
  - **Expected return** is \(0.0005\), that is \(0.5\%\).

- Would you bet everything on PHARMA with these cards? **something is missing in our formulation**
Portfolio optimization needs to input the investor’s aversion to risk.

Using $x = \text{argmax}(\mathbb{E}(\rho(x)))$ can lead the investor to forget about risk.

Solution: include risk in the program. Risk is vaguely a quantification of the dispersion of the returns of a portfolio.

Different choices:

- **Variance**:
  - $C$ is the covariance matrix of the vector r.v. $R$ takes values in $\mathbb{R}^n$,
    $$ C = \mathbb{E}[(R - \mathbb{E}[R])(R - \mathbb{E}[R])^T]. $$
  - The variance of $\rho(x)$ is simply $x^T C x$.
  - Maximal expected return under variance constraints = mean-variance optimization.

- **Mean-absolute deviation (MAD)**:
  - Namely $\mathbb{E}[|\rho(x) - \mathbb{E}[\rho(x)]|] = E[|x^T \bar{R}|]$ where $\bar{R} = R - \mathbb{E}[R]$.
  - Penalized estimation: $x = \text{argmax}_{x \geq 0, x^T 1_n = 1} \lambda \cdot \mathbb{E}[\rho(x)] - \mathbb{E}[|x^T \bar{R}|]$. 

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Risk

- The **variance** formulation leads to a quadratic program:

\[
\begin{align*}
\text{maximize} & \quad x^T \mathbb{E}[R] \\
\text{subject to} & \quad x \geq 0, \ x^T 1_n = 1 \\
& \quad x^T C x \leq \lambda
\end{align*}
\]

- The **MAD** formulation leads to something closer to linear programming:

\[
\begin{align*}
\text{maximize} & \quad \lambda x^T \mathbb{E}[R] - \mathbb{E}[|x^T \bar{R}|] \\
\text{subject to} & \quad x \geq 0, \ x^T 1_n = 1
\end{align*}
\]

- **Problem**: lots of expectations $\mathbb{E}...$

- We need to fill in some expected values above by some guesses.
Approximations

• We write \( \tilde{r} \) for \( \mathbb{E}[R] \) which can be guessed according to...
  - research, analysts playing with excel, valuation models.
  - historical returns.

• We also need to approximate \( \mathbb{E}[|x^T \tilde{R}|] \).

• Suppose we have a history of \( N \) returns \( (r^1, \cdots, r^N) \) where each \( r \in \mathbb{R}^n \).
  - Write \( \bar{r} = \sum_{j=1}^{N} r^j \).
  - in practice, approximate \( \mathbb{E}[|x^T \tilde{R}|] \approx \sum_{j=1}^{N} |x^T (r^j - \bar{r})| \)

• this becomes:

\[
\begin{align*}
\text{maximize} & \quad \lambda x^T r - \frac{1}{N} \sum_{j=1}^{N} |x^T (r^j - \bar{r})| \\
\text{subject to} & \quad x \geq 0, \ x^T 1_n = 1
\end{align*}
\]

• Now add artificial variables \( y_j = |x^T (r^j - \bar{r})| \). One for each observation. Now,

\[
\begin{align*}
\text{maximize} & \quad \lambda x^T r - \frac{1}{N} \sum_{j=1}^{N} y_j \\
\text{subject to} & \quad x \geq 0, \ y_j \geq 0, \ x^T 1_n = 1, \\
& \quad -y_j \leq x^T (r^j - \bar{r}) \leq y_j, \ j = 1, \cdots, N
\end{align*}
\]
LP’s, Duality and Arbitrage
Duality and Arbitrage

- We propose in this an economic interpretation of duality
- Due to Arrow, Debreu, in the 50’s...
- Used every day on financial markets (sometimes unknowingly)
- Simple LP duality result, but underpins most of modern finance theory...
One period model

- As in the previous section, basic discrete, one period model on a single asset.
- Its price today is $q_1$. Its (random) price time $T$ ahead is $x$.
- Assume $x$ can only take any of the following values

$$x \in \{x_1, \ldots, x_n\}$$

at a maturity date $T$, and that we have an estimate of their probabilities,

$$\{p_1, \ldots, p_n\}.$$ 

- We have discretized the space of possibilities.
- We can only trade today and at maturity
- There is a cash security worth $1$ today, that pays $1$ at maturity
- near-zero interest rates. sounds familiar?
One period model

- There are also $m - 1$ other securities with payoffs at maturity given by

  $$h_k(x_i) \quad \text{if } x = x_i \text{ at time } T$$

  for $k = 2, \ldots, m - 1$.

- The payoffs are arbitrary functions of the $n$ possible values of the asset at time $T$.

- We could have $h_k(x) = x^2$. Or that for $i \leq j$, $h_k(x_i) = 0$, $i > j$, $h_k(x_i) = 1$.

- We denote by $q_k$ the price today of security $k$ with payoff $h_k(x)$.

All these securities are tradeable, can we use them to get information on the price of another security with payoff $h_0(x)$?
Static Arbitrage

Remember:

- We can only trade today and at maturity.
- We can only trade in securities which are priced by the market.

We want to exclude arbitrage strategies

- If the payoff of a portfolio $A$ is always larger than that of a portfolio $B$ then $\text{Price}(A) \geq \text{Price}(B)$.
- The price of the sum of two products is equal to the sum of the prices.
Simplest Example: Put Call Parity

\[ \text{Payoff} \]

\[ K - S \]

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Price bounds

Suppose that we form a portfolio of cash, stocks and securities $h_k(x)$ with coefficients $\lambda_k$:

\[
\begin{align*}
\lambda_0 & \quad \text{in cash} \\
\lambda_1 & \quad \text{in stock} \\
\lambda_k & \quad \text{in security } h_k(x)
\end{align*}
\]

- All portfolios that satisfy

\[
\lambda_0 + \lambda_1 x_i + \sum_{k=2}^{m} \lambda_k h_k(x_i) \geq h_0(x_i) \quad i=1, \ldots, n
\]

must be more expensive than the security $h_0(x)$

- All portfolios that satisfy the opposite inequality must be cheaper

- For portfolios that satisfy neither of these, nothing can be said.

- We are just comparing portfolios dominated for all outcomes of $x$. 

Price bounds

• For each of these portfolios, we get an upper/lower bound on the price today of the security $h_0(x)$.

• We can look for optimal bounds...

• We can solve:

  \[
  \begin{align*}
  \text{minimize} & \quad \lambda_0 + \lambda_1 q_1 + \sum_{k=1}^{m} \lambda_k q_k \\
  \text{subject to} & \quad \lambda_0 + \lambda_1 x_i + \sum_{k=2}^{m} \lambda_k h_k(x_i) \geq h_0(x_i), \quad i = 1, \ldots, n
  \end{align*}
  \]

  • Linear program in the variable $\lambda \in \mathbb{R}^{(m+1)}$
  • Produces an optimal upper bound on the price today of the security $h_0(x)$
The original linear program looks like:

\[
\begin{align*}
\text{minimize} & \quad c^T \lambda \\
\text{subject to} & \quad A\lambda \geq b
\end{align*}
\]

which is a linear program in the variable \( \lambda \in \mathbb{R}^m \).

We can form the Lagrangian

\[ L(\lambda, p) = c^T \lambda + y^T (b - A\lambda) \]

in the variables \( \lambda \in \mathbb{R}^m \) and \( y \in \mathbb{R}^n \), with \( y \succeq 0 \).
Linear Programming Duality

- We then minimize in $\lambda$ to get the dual function

$$g(y) = \inf \limits_{\lambda} c^T \lambda + y^T (b - A\lambda)$$

for $y \succeq 0$, which is again

$$g(y) = \inf \limits_{\lambda} y^T b + \lambda^T (c - A^T y)$$

and we get:

$$g(y) = \begin{cases} y^T b & \text{if } c - A^T y = 0 \\ -\infty & \text{if not.} \end{cases}$$
Linear Programming Duality

• With

\[ g(y) = \begin{cases} 
  y^T b & \text{if } c - A^T y = 0 \\
  -\infty & \text{if not.}
\end{cases} \]

• we get the dual linear program as:

maximize \[ b^T y \]
subject to \[ A^T y = c \]
\[ y \geq 0 \]

which is also a linear program in \( x \in \mathbb{R}^n \).
LP duality: summary

• The primal LP is the original linear program looks like:

\[
\begin{align*}
\text{minimize} & \quad c^T \lambda \\
\text{subject to} & \quad A\lambda \geq b
\end{align*}
\]

• its dual is then given by:

\[
\begin{align*}
\text{maximize} & \quad b^T y \\
\text{subject to} & \quad A^T y = c \\
& \quad y \geq 0
\end{align*}
\]

**Strong duality**: both optimal values are equal
Let’s look at what this produces for the portfolio problem. . .

- The **primal** problem in the variable $\lambda \in \mathbb{R}^m$ is given by:

  $$\begin{align*}
p^{\text{max}} := \min & \quad \lambda_0 + \lambda_1 q_1 + \sum_{k=2}^{m} \lambda_k q_k \\
\text{s.t.} & \quad \lambda_0 + \lambda_1 x_i + \sum_{k=2}^{m} \lambda_k h_k(x_i) \geq h_0(x_i), \quad i = 1, \ldots, n
\end{align*}$$

- The **dual** in the variable $y \in \mathbb{R}^n$ is then

  $$\begin{align*}
p^{\text{max}} := \max & \quad \sum_{i=1}^{n} y_i h_0(x_i) \\
\text{s.t.} & \quad \sum_{i=1}^{n} y_i h_k(x_i) = q_k, \quad k = 2, \ldots, m \\
& \quad \sum_{i=1}^{n} y_i x_i = q_1 \\
& \quad \sum_{i=1}^{n} y_i = 1 \\
& \quad y \geq 0
\end{align*}$$
The last two constraints \( \{ \sum_{i=1}^{n} y_i = 1, \; y \geq 0 \} \) mean that \( y \) is a \textit{probability measure}.

We can rewrite the previous program as:

\[
p_{\text{max}} := \max \quad \mathbb{E}_y[h_0(x)] \\
\text{s.t.} \quad \mathbb{E}_y[h_k(x)] = q_k, \quad k = 2, \ldots, m \\
\mathbb{E}_y[x] = q_1
\]

\( y \) is a \textit{probability measure}.

We can compute \( p_{\text{min}} \) by minimizing instead.
LP duality & arbitrage

- What does this mean?

- There are three ranges of prices for the security with payoff $h_0(x)$:
  - Prices above $p_{\text{max}}$: these are not viable, you can get a cheaper portfolio with a payoff that always dominates $h_0(x)$.
  - Prices in $[p_{\text{min}}, p_{\text{max}}]$: prices are viable, i.e. compatible with the absence of arbitrage.
  - Prices below $p_{\text{min}}$: these are not viable, you can get a portfolio that is more expensive than $h_0(x)$ with a payoff that is always dominated by $h_0(x)$. 
Price bounds

- Example:
  - Suppose the product in the objective is a call option:
    \[ h_0(x) = (x - K)^+ \]
    where \( K \) is called the strike price.
  - Suppose also that we know the prices of some other instruments
  - We get upper and lower price bounds on the price of this call for each strike \( K \)

- On a graphic...
Price Bounds

![Graph showing price bounds with arbitrage and model prices labeled.](image)
LP duality & arbitrage

• What if there is no solution $y$ and the linear program is infeasible?
  ○ Then the original data set $q$ must contain an arbitrage.
  ○ Start with one product, stock and cash... and test.
  ○ Increase the number of products...
Fundamental theorem of asset pricing

**Theorem 1.** In the one period model, there is no arbitrage between the prices \( \{q_0, \ldots, q_m\} \) of securities with payoffs at maturity \( \{h_0(x), \ldots, h_m(x)\} \)

\[
\uparrow
\]

There exists a probability \( y \) (with \( \sum_{i=1}^{n} y_i = 1 \) and \( y \geq 0 \)) such that

\[
q_k = E_y[h_k(x)], \quad k = 0, \ldots, m
\]
- Because prices are computed using *expectations under* $y$ (and not expected utility/certain equivalent), we call the probability $y$ risk-neutral.

- In particular, it satisfies $q_1 = \mathbb{E}_y[x]$.

- If there are *constant* interest rates, simply use discounted values for *prices at maturity*.

- This probability $y$ has *nothing to do* with the observed distribution of the asset $x$ or its past distribution! (Very common mistake)
Because one can trade
  - the asset
  - derivative products based on the asset
to form portfolios to hedge/replicate other products, it is possible to evaluate these products using expected value under an appropriate choice of probability.

Again, the risk-neutral probability $y$ is a tool inferred from market prices,
it has nothing to do with the statistical properties of the underlying asset $x$.

Linear programming duality is interpreted as a duality between portfolios on assets problems and probabilities (models)
LP duality & arbitrage

In the previous result:

- Set of possible \textbf{probabilistic models} = \textbf{probability simplex}:
  \[ p_i \geq 0, \quad \sum_i p_i = 1 \]

- Expected value, hence price is linear in the probability \( p_i \)
  \[
  \mathbb{E}[h(x)] = \sum_i p_i h(x_i)
  \]

- A price constraint is just a linear equality constraint on the probabilities:
  \[
  \sum_i p_i h(x_i) = b_i
  \]

- Simple family of distributions.
Moment constraints

Choices for asset pricing formulas that depend on the prices directly:

- Use indicator function as payoff:

\[ h(x) = 1_{\{x \geq K\}} \]

This produces the constraint:

\[ \sum_i p_i 1_{\{x_i \geq K\}} = P(X \geq K) = b \]

- Also, quadratic variation:

\[ h(x) = x^2 \]

Corresponds to:

\[ \sum_i p_i x_i^2 = \mathbb{E}[x_i^2] = b \]
Moment constraints

Higher order formulations? Variance?

- We can’t incorporate a variance swap
- A constraint of the form
  \[ \text{Variance}(x) = qV \]
  why?
- Becomes \( \sum_i p_i x_i^2 - \left( \sum_i p_i x_i \right)^2 = qV \Rightarrow \) quadratic constraints in \( p_i \).
- Would however works if we also fix the expected value:
  \[ \mathbb{E}[x] = b \]

Corresponds to a \textbf{forward} price (EV of the asset):

\[ \sum_i p_i x_i = q_F \quad \text{and} \quad \text{Variance}(x) = \sum_i p_i x_i^2 - q_F^2 = qV \]

- We came back to a simple \textbf{linear constraint}
Option price vs. variance

• Fix the forward price (expected value of the asset), move the variance. . .
• We study the price of a call option \( h_0 \).

\[
\begin{align*}
\text{maximize} & \quad \sum_i p_i \, h_0(x_i) \\
\text{subject to} & \quad \sum_i p_i \, x_i = S_0 \\
& \quad \sum_i p_i \, x_i^2 = b^2 \\
& \quad 0 \leq p_i \leq 1,
\end{align*}
\]

• Look at the price as a function of \( b^2 \) . . .
Option pricing & LP: example
Option pricing

Option pricing example. . .

• Study the price **CutCall** option, with payoff:

\[ h_0(X) = (X - K)^+ 1_{\{X \leq L\}} \]

• Similar to knock-out option but only **check at maturity**. No knock-out during its life, **European** kind of knock-out.

• Get some market prices \( q_k \) for **regular** calls:

\[ h_k(X) = (X - K_k)^+ \]

• Solve for the maximum CutCall price:

\[
\begin{align*}
\text{maximize} & \quad \sum_i p_i h_0(x_i) \\
\text{subject to} & \quad \sum_i p_i h_k(x_i) = q_k \\
& \quad \sum_i p_i = 1 \\
& \quad p_i \geq 0
\end{align*}
\]
Option pricing

Solve

\[
\begin{align*}
\text{maximize} & \quad \sum_i p_i h_0(x_i) \\
\text{subject to} & \quad \sum_i p_i h_k(x_i) = q_k \\
& \quad \sum_i p_i = 1 \\
& \quad p_i \geq 0
\end{align*}
\]

with

\[K = \{50, 80, 110, 120, 150, 280\}\]

and vector of prices for the 6 options.

\[q = (102.9167, 79.5667, 59.2167, 53.1000, 36.7500, 0.5667)\]

- Prices were computed above using the uniform distribution on \([0, 300]\)
- **Result:** maximum price for the CutCall is \textbf{59}
- Next slide: risk neutral distribution for that maximal price.
Corresponding Risk-Neutral Probability

\[ \frac{p_i}{X} \]

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Option pricing

- Problem in dimension 2, price a \textbf{basket options} with payoff

\[(x_1 + x_2 - K)_+\]

- The input data set is composed of the asset prices together with the following call prices:

\[(.2x_1 + x_2 - .1)_+, \ (.5x_1 + .8x_2 - .8)_+, \]
\[(.5x_1 + .3x_2 - .4)_+, \ (x_1 + .3x_2 - .5)_+, \]
\[(x_1 + .5x_2 - .5)_+, \ (x_1 + .4x_2 - 1)_+, \]
\[(x_1 + .6x_2 - 1.2)_+.\]
Option pricing

Price bounds

strike

price

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Run another test:

- Look at how these bounds evolve as more and more instruments are incorporated into the data set.
- Fix $K = 1$, we compute the bounds using only the $k$ first instruments in the data set, for $k = 2, \ldots, 7$.
- Plot the upper and lower bounds
- Also plot one of the solutions

Conclusion: more market values $\Rightarrow$ tighter bounds
Option pricing
Figure 1: Example of discrete distribution minimizing the price of $(x_1 + x_2 - K)_+$. 
Caveats

Size!

- Grows \textbf{exponentially} in $k^n$ with the number of points
- Only works with \textit{discrete} and \textit{bounded} models

Everything comes at a price. . .