Reminder: Basic Feasible Solutions, Extreme points, Optima

Three fundamental theorems:

• Let $x$ be a basic feasible solution (BFS) to a LP with index set $I$ and objective value $z$. If $\exists e, 1 \leq e \leq n, e \notin I$ such that $c_e - z_e > 0$ and at least one $y_{i,e} > 0$, then we can have a better basic feasible solution by replacing an index in $I$ by $e$ with a new objective $\hat{z} \geq z$, strictly if $x_I$ is non-degenerate.

• Let $x^*$ be a basic feasible solution (BFS) to a LP with index set $I$ and objective value $z^*$. If $c_i - z^*_i \leq 0$ for all $1 \leq i \leq n$ then $x^*$ is optimal.

• Let $x$ be a basic feasible solution (BFS) to a LP with index set $I$. If $\exists$ an index $e \notin I$ such that $y_e \leq 0$ then the feasible region is unbounded. If moreover for $e$ the reduced cost $c_e - z_e > 0$ then there exists a feasible solution with at most $m + 1$ nonzero variables and an arbitrary large objective function.
Today

- Visualizing the simplex.
- Example of tableaux in canonical feasible form.
Reflecting on the Algorithm
So far, what is the simplex?

- The simplex is a family of algorithms which do the following:

  1. Obtains an initial Basic feasible solution. more on that later.
  2. iterates: move from one BFS $I$ to a better BFS $I'$:
     - check reduced cost coefficients $c_j - c_I^T B_I^{-1} a_j, j \in O$. if all negative $I$ is optimal, OVER.
     - otherwise, pick one index $e$ for which it is positive. this will enter $I$.
     - Check coordinates of $y_e = B_I^{-1} a_e$. if all $\leq 0$ then optimum is unbounded, OVER.
     - otherwise, take the index $r$ such that it achieves the minimum in $
      \left\{ \frac{x_{ij,e}}{y_{j,e}} \big| y_{j,e} > 0, 1 \leq j \leq m \right\}$, this will ensure feasibility. The $r$th index of the base $I$ is $i_r \leq n$.
     - $I' = \{ I \setminus i_r \} \cup e$.
     - We have improved on the objective. If $x_I$ was not degenerate, we have strictly improved.
     - $I \leftarrow I'$

- The loop is on a finite set of extreme points. it either exits early (unbounded), exits giving an answer (optimum $I^*$ and corresponding solution $x^*$) or loops indefinitely (degeneracy).
A Matlab Demo With Polyhedrons Containing the Origin
A Matlab Demo With Polyhedrons Containing the Origin

now with the real matlab demo...
Tableaux with Canonical Feasible Form
WHY tableaux?

- Last time: an example where we move from a base $\mathbf{I}$ to a new base $\mathbf{I}'$, compute $B_{I'}^{-1}$, do the multiplications etc. and reach the optimum. This is the simplex.

- **Double issue:**
  - **Computational 1:** inverting matrices costs time & money. One column is different between $B_\mathbf{I}$ and $B_{I'}$, can we do better than inverting everything again?
  - **Computational 2:** multiplying matrices costs time & money. $B_\mathbf{I}^{-1}A$ and $B_{I'}^{-1}A$ are related.
WHY tableaux?

- **Down to what we really need at each iteration:**
  - reduced cost coefficients vector \((c_i - z_i)\) of \(\mathbf{R}^n\) to pick an index \(e\) and check optimality,
  - All column vectors of \(A\) in the base \(\mathbf{I}\), that is \(Y\), to check boundedness and choose \(r\), namely all coordinates of \(y_e = B_\mathbf{I}^{-1}a_e\) in particular.
  - The current basic solution vector, \(B_\mathbf{I}^{-1}b\) both to choose \(r\) and on exit.
  - Having also the objective \(c_\mathbf{I}^T B_\mathbf{I}^{-1}b\) would help.

- **Summing up,** we need something that keeps track of

\[
\begin{array}{c|c}
\vdots & \mathbf{B}_\mathbf{I}^{-1}A \\
\vdots & \vdots \\
(c - z)' & \vdots \\
\end{array}
\begin{array}{c|c}
\mathbf{B}_\mathbf{I}^{-1}b \\
\vdots \\
c_\mathbf{I}^T \mathbf{B}_\mathbf{I}^{-1}b \\
\end{array}
\]
Canonical Feasible Form: We know an initial BFS to corresponding Standard Form

- Let's **standardize** a feasible (i.e. \( b \geq 0 \)) canonical form:

  \[
  \begin{align*}
  \text{maximize} & \quad \alpha^T u \\
  \text{subject to} & \quad Mu \leq b \\
  & \quad u \geq 0
  \end{align*}
  \]

- We assume that \( u, \alpha \in \mathbb{R}^d \) for a \( d \) dimensional objective and \( M \in \mathbb{R}^{m \times d} \) and \( b \in \mathbb{R}^m \) for \( m \) constraints.
Canonical Feasible Form: We know an initial BFS to corresponding Standard Form

- Slack variables $x_{d+1}, \cdots, x_{d+m}$ can be added so that $[A, I_m] \begin{bmatrix} x_{d+1} \\ \vdots \\ x_{d+m} \end{bmatrix} = b$ and the problem is now with $c = [\alpha, 0, \cdots, 0] \in \mathbb{R}^{d+m}$

$$\text{maximize} \quad x_0 = c^T x$$
$$\text{subject to} \quad \begin{cases} [M, I_m]x &= b \\ x &\geq 0 \end{cases}$$

- $x, c \in \mathbb{R}^{m+d}, \ c = [\alpha, 0], \ A = [M, I_m] \in \mathbb{R}^{m \times (m+d)}$ and same $b \in \mathbb{R}^m$.

- The dimensionality of the problem is now $n = d + m$. 
## Simplex Method: Tableau

Let us represent this by an (annotated) tableau:

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\cdots$</th>
<th>$x_e$</th>
<th>$\cdots$</th>
<th>$x_d$</th>
<th>$x_{d+1}$</th>
<th>$x_{d+2}$</th>
<th>$\cdots$</th>
<th>$x_{d+r}$</th>
<th>$\cdots$</th>
<th>$x_{d+m}$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{d+1}$</td>
<td>$m_{11}$</td>
<td>$m_{12}$</td>
<td>$\cdots$</td>
<td>$m_{1e}$</td>
<td>$\cdots$</td>
<td>$m_{1d}$</td>
<td>1</td>
<td>0</td>
<td>$\cdots$</td>
<td>0</td>
<td>$\cdots$</td>
<td>0</td>
<td>$b_1$</td>
</tr>
<tr>
<td>$x_{d+2}$</td>
<td>$m_{21}$</td>
<td>$m_{22}$</td>
<td>$\cdots$</td>
<td>$m_{2e}$</td>
<td>$\cdots$</td>
<td>$m_{2d}$</td>
<td>0</td>
<td>1</td>
<td>$\cdots$</td>
<td>0</td>
<td>$\cdots$</td>
<td>0</td>
<td>$b_2$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\ddots$</td>
<td>$\vdots$</td>
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<td>$\ddots$</td>
<td>$\vdots$</td>
<td>$\ddots$</td>
<td>$\ddots$</td>
<td></td>
</tr>
<tr>
<td>$x_{d+r}$</td>
<td>$m_{r1}$</td>
<td>$m_{r2}$</td>
<td>$\cdots$</td>
<td>$m_{re}$</td>
<td>$\cdots$</td>
<td>$m_{rd}$</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
<td>1</td>
<td>$\cdots$</td>
<td>0</td>
<td>$b_r$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\ddots$</td>
<td>$\vdots$</td>
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<td>$\vdots$</td>
<td>$\ddots$</td>
<td>$\ddots$</td>
<td></td>
</tr>
<tr>
<td>$x_{d+m}$</td>
<td>$m_{m1}$</td>
<td>$m_{m2}$</td>
<td>$\cdots$</td>
<td>$m_{me}$</td>
<td>$\cdots$</td>
<td>$m_{md}$</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
<td>0</td>
<td>$\cdots$</td>
<td>1</td>
<td>$b_m$</td>
</tr>
<tr>
<td><em>$x_0$</em></td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$\cdots$</td>
<td>$c_e$</td>
<td>$\cdots$</td>
<td>$c_d$</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
<td>0</td>
<td>$\cdots$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- Since $b \geq 0$, take an original BFS as $\begin{bmatrix} 0, \cdots, 0, b_1, b_2, \cdots, b_m \end{bmatrix}^T$
- Why:
  - **basic**: $I = \{d + 1, \ldots, d + m\}$
  - **feasible**: $[0, \cdots, 0, b_1, b_2, \cdots, b_m]^T \geq 0$. 

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Simplex Method: Tableau

- the structure of the tableau so far,

<table>
<thead>
<tr>
<th>A</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>c^T</td>
<td>0</td>
</tr>
</tbody>
</table>

- The index set \( I \) so far \( \{d + 1, d + 2, \cdots , d + m\} \).

\[ B_I = I_m, \quad B_I^{-1}b = b, \quad B_I^{-1}A = A \text{ etc.} \]

- The lower-right coincides with the objective so far... 0

- \( c \) is actually equal to \( (c - z_I) \) when \( I \) only describes slack variables.
Simplex Method without non-negativity and objectives...

- Remember: a basis $I$ gives a sparse solution $x_I$.  
- **there’s one basis** $I^*$ which is the good one.  
- The solution is $x$ such that $x^*_I = B^{-1}_{I^*}b$ and the rest is zero.  
- We can start with the **slack variables** as a basis in canonical feasible form.  
- Under this form, the first matrix basis is $B_I = I_m$ the identity matrix.  
- We will **move** from one basis to the other. We’ve proved this is possible.  
- In doing so, we also have to recast the cost.  
- Let’s check how it looks in practice, without looking at feasibility and objective related concepts.
Consider now taking a variable out of $I$ to replace it by a variable in $O$.

The $r$th index of $I$, $i_r$, leaves the basis, $e$ initially in $O$ is removed.
Two ways of looking at the same operation:

○ Through elementary row/column operations transfer a vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ where the 1 is in position $r$ to get a similar basis vector in the $e$th column of $A$.

\[
\begin{bmatrix}
\vdots & i_r & \cdots & e & \cdots \\
\vdots & 0 & \cdots & \vdots & \cdots \\
r & 0 & \cdots & a_{re} & \cdots \\
i & 0 & \cdots & a_{ie} & \cdots \\
\vdots & 0 & \cdots & \vdots & \cdots \\
\end{bmatrix} \Rightarrow \begin{bmatrix}
\vdots & \cdots & i_r & \cdots & e & \cdots \\
\vdots & \cdots & \cdots & 0 & \cdots \\
r & \cdots & a_{r,i_r} & \cdots & 1 & \cdots \\
i & \cdots & a_{i,i_r} & \cdots & 0 & \cdots \\
\vdots & \cdots & \cdots & 0 & \cdots \\
\end{bmatrix}
\]
Consider the equalities $Ax = b$ written in row form,

$$u_i^T x = b_i$$

where the $u$’s are the rows of $A$.

Putting variable $x_e$ in the basis is equivalent to isolating $x_e$ so that it is present in **all but one** of the $m$ equations, with coefficient 1. On the other hand we let $x_{ir}$ enter all equations again, that is

$$x_e = \tilde{b}_i - \sum_{i=1, i \neq e}^{n} \rho_i x_i$$

and $x_e$ does not appear elsewhere.
This is achieved through a pivot in the tableau.

Once the $r$th element of basis $I$, namely column $i_r \leq n$, and $e \leq n$ are agreed upon, the rules to update the tableau are:

(a) in pivot row $a_{rj} \leftarrow a_{rj}/a_{re}$.

(b) in pivot column $a_{re} \leftarrow 1$, $a_{ie} = 0$ for $i = 1, \cdots, m$, $i \neq r$: the $e$th column becomes a matrix of zeros and a one.

(c) for all other elements $a_{ij} \leftarrow a_{ij} - \frac{a_{rj}a_{ie}}{a_{re}}$. 
**The Gauss pivot**

- **Graphically,**

\[
\begin{bmatrix}
\cdots & j & \cdots & e & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
r & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
\cdots & j & \cdots & e & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots \\
r & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

- **Look at how the column** \( e \) **is now a column of 0 and 1’s. This makes sense since**

\[
B_I^{-1} a_e = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}
\text{ with 1 in } e^{th} \text{ position means } a_e \text{ is in the basis.}
\]
• Consider the linear system

\[
\begin{align*}
    x_1 + x_2 - x_3 + x_4 &= 5 \\
    2x_1 - 3x_2 + x_3 + x_5 &= 3 \\
    -x_1 + 2x_2 - x_3 + x_6 &= 1
\end{align*}
\]

• The corresponding tableau

\[
\begin{bmatrix}
    a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b \\
    1 & 1 & -1 & 1 & 0 & 0 & 5 \\
    2 & -3 & 1 & 0 & 1 & 0 & 3 \\
    -1 & 2 & -1 & 0 & 0 & 1 & 1
\end{bmatrix}
\]
Simplex Method: Swapping Indexes

- in the corresponding tableau,

\[
\begin{bmatrix}
a_4 & 1 & 1 & -1 & 1 & 0 & 0 & 5 \\
a_5 & 2 & -3 & 1 & 0 & 1 & 0 & 3 \\
a_6 & -1 & 2 & -1 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

notice the structure:

\[
\begin{array}{cccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\vdots & M & \vdots & I_3 & \vdots & b \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\end{array}
\]

- And the fact that by taking the obvious basis \( I = \{4, 5, 6\} \) we have \( B_I = I_3 \) and \( B_I^{-1} = I_3 \)
Simplex Method: Let’s pivot

- Let’s pivot arbitrarily. We put 1 in the base and remove 4.

\[
\begin{bmatrix}
\begin{array}{cccccc}
x_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b \\
a_4 & 1 & 1 & -1 & 1 & 0 & 0 & 5 \\
a_5 & 2 & -3 & 1 & 0 & 1 & 0 & 3 \\
a_6 & -1 & 2 & -1 & 0 & 0 & 1 & 1 \\
\end{array}
\end{bmatrix}
\]

which yields

\[
\begin{bmatrix}
\begin{array}{cccccc}
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b \\
a_1 & 1 & 1 & -1 & 1 & 0 & 0 & 5 \\
a_5 & 0 & -5 & 3 & -2 & 1 & 0 & -7 \\
a_6 & 0 & 3 & -2 & 1 & 0 & 1 & 6 \\
\end{array}
\end{bmatrix}
\]

- \( I = \{1, 5, 6\} \), that is \( B_I = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} \). The basic solution is such that \( x_I = B_I^{-1} b \)

- Note that all coordinates of \( a_1, \cdots, a_6, b \) in the table are given with respect to \( a_1, a_5, a_6 \). In particular the last column corresponds to \( B_I^{-1} b \)...not feasible here BTW.
Simplex Method: again…

- Let’s pivot arbitrarily again, this time inserting 2 and removing the second variable of the basis, 5.

\[
\begin{bmatrix}
1 & 1 & -1 & 1 & 0 & 0 & 5 \\
0 & -5 & 3 & -2 & 1 & 0 & -7 \\
0 & 3 & -2 & 1 & 0 & 1 & 6 \\
\end{bmatrix}
\]

- Notice how one can keep track of who is in the basis by checking where 0/1’s columns are.

- The solution is now feasible... pure luck.
Simplex Method: and again...

- once again, pivot inserting $3$ and removing the \textbf{third} variable of the basis, $6$.

\[ \begin{bmatrix}
  a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b \\
  1 & 0 & -\frac{2}{3} & \frac{3}{5} & \frac{1}{5} & 0 & \frac{18}{5} \\
  0 & 1 & -\frac{1}{5} & -\frac{1}{5} & \frac{3}{5} & 1 & \frac{7}{5} \\
  0 & 0 & -\frac{1}{5} & \frac{2}{5} & -1 & 0 & 18 \\
\end{bmatrix} \]

- horrible. moving randomly we have a now non-feasible degenerate basic solution.

- yet we knew that pivoting randomly based only on $y_{r,e} \neq 0$ would lead us nowhere.
Adding the reduced costs

• What happens when we also pivot the last line?

• Remember the last line is equal to $v \overset{\text{def}}{=} (c - z)'$ in the beginning.

• Remember also that

  (a) in pivot row $a_{rj} \leftarrow a_{rj}/a_{re}$.
  (b) in pivot column $a_{re} \leftarrow 1$, $a_{ie} = 0$ for $i = 1, \ldots, m, i \neq r$: the $e$th column becomes a matrix of zeros and a one.
  (c) for all other elements $a_{ij} \leftarrow a_{ij} - \frac{a_{rj}a_{ie}}{a_{re}}$

• Here, (a) does not apply, we cannot be in the pivot row.

• we have

  ◦ in pivot column $v_e = 0$: makes sense, reduced cost is zero for basis elements.
  ◦ for all other elements $v_j \leftarrow v_j - \frac{a_{rj}v_e}{a_{re}}$. 

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Adding the reduced costs

- Recapitulating, at each iteration of the pivot the matrix is exactly

\[
\begin{array}{c}
\vdots & \vdots & \vdots & \vdots & \vdots \\
B_I^{-1}M & \vdots & B_I^{-1} & \vdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\begin{array}{c}
\vdots \\
B_I^{-1}b \\
\vdots \\
-x_0 \\
\end{array}
\]

- The pivot is thus applied on the \( m + 1 \times n + 1 \) tableau.

- The tableau contains **everything we need**, reduced costs, (minus)objective, the coordinates of \( B_I^{-1}b \) and \( B_I^{-1}A \)
Tableaux with Arbitrary Initial BFS
• Suppose that we are given an arbitrary BFS $I$ for the problem

$$\text{maximize} \quad x_0 = c^T x$$

subject to

$$\begin{cases} 
Ax &= b \\
x &\geq 0
\end{cases}$$

• We try to go back to the previous situation.
Working around to go back to previous situation

- Perform a permutation of columns such that columns in positions \( i_1, \cdots, i_m \) become in last positions \( n - m + 1, \cdots, n \).

- \( A \) is now \([N, B]\) (N for non-basic part) and the system can be written as

\[
\begin{align*}
N x_N + B x_B &= b \\
 c_N^T x_N + c_B^T x_B &= x_0
\end{align*}
\]

- Multiplying the first line by \( B^{-1} \),

\[
B^{-1} N x_N + x_B = B^{-1} b \quad \text{thus} \quad x_B = B^{-1} b - B^{-1} N x_N
\]

which when used for objective \( x_0 \) yields

\[
x_0 = \left( c_N - c_B^T B^{-1} N \right)^T x_N + c_b^T B^{-1} b
\]
Working around to go back to previous situation

• We can now use the same tableau:

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
: & B^{-1}N & : & I & : & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & c^T_N - z^T_N & \ldots & 0 & \ldots & -x_0 \\
\end{array}
\]

• And can apply the simplex as defined for canonical feasible forms.
Short Comment on Dictionaries

- A dictionary is a comparable compact form

\[
\begin{align*}
\text{maximize} & \quad \zeta = 4x_2 + 3x_3 \\
\text{subject to} & \quad w_1 = 5 - 2x_1 - 3x_2 - x_3 \\
& \quad w_2 = 11 - 4x_1 - x_2 - 2x_3 \\
& \quad w_3 = 8 - 3x_1 - 4x_2 - 2x_3 \\
& \quad x_1, x_2, x_3, w_1, w_2, w_3 \geq 0.
\end{align*}
\]

where basic variables are kept on the top and non-basic are kept on the left.

- We save space (1/0 columns) but need to keep track of variable names.
- The constants on the left correspond to the last column in tableaux.
- The first line stands for reduced cost coefficients of nonbasic variables.
- The lower-right corresponds to minus the $B_I^{-1}A$ matrix for indices in $O$.
- Equivalent to Tableaux, rather used for educational purposes.