ORF 522

Linear Programming and Convex Analysis

The Simplex Method, Tableaux and Dictionaries

Marco Cuturi

Reminder: Basic Feasible Solutions, Extreme points, Optima

Three fundamental theorems:

- Let $\mathbf{x}$ be a **basic feasible solution (BFS)** to a LP with index set I and objective value z. If $\exists e, 1 \leq e \leq n, \boldsymbol{e \notin I}$ such that $c_e - \boldsymbol{z_e} > 0$ **and** at least one $\boldsymbol{y_{i,e} > 0}$, then we can have a **better basic feasible** solution by replacing an index in I by e with a new objective $\hat{z} \geq z$, **strictly** if x_I is non-degenerate.

- Let $\mathbf{x}^\star$ be a **basic feasible solution (BFS)** to a LP with index set I and objective value $\boldsymbol{z^\star}$. If $c_i - \boldsymbol{z_i^\star} \leq 0$ for all $1 \leq i \leq n$ then $\mathbf{x}^\star$ is optimal.

- Let $\mathbf{x}$ be a **basic feasible solution (BFS)** to a LP with index set I. If $\exists$ an index $e \notin \mathbf{I}$ such that $\mathbf{y}_e \leq 0$ then the **feasible region** is **unbounded**. If moreover for e the reduced cost $c_e - z_e > 0$ then there exists a feasible solution with at most $\mathbf{m+1}$ nonzero variables and an **arbitrary large objective function**.

Today

- Visualizing the simplex.
- Example of tableaux in canonical feasible form.

Reflecting on the Algorithm

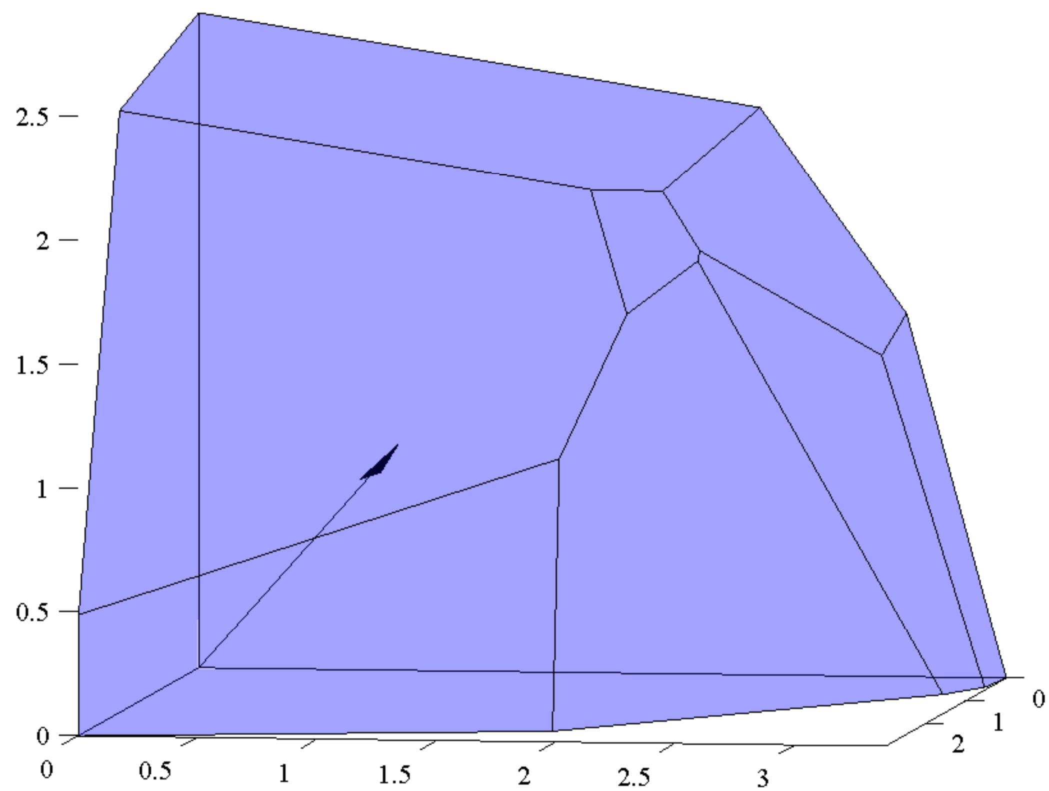
So far, what is the simplex?

- The simplex is a family of algorithms which do the following:

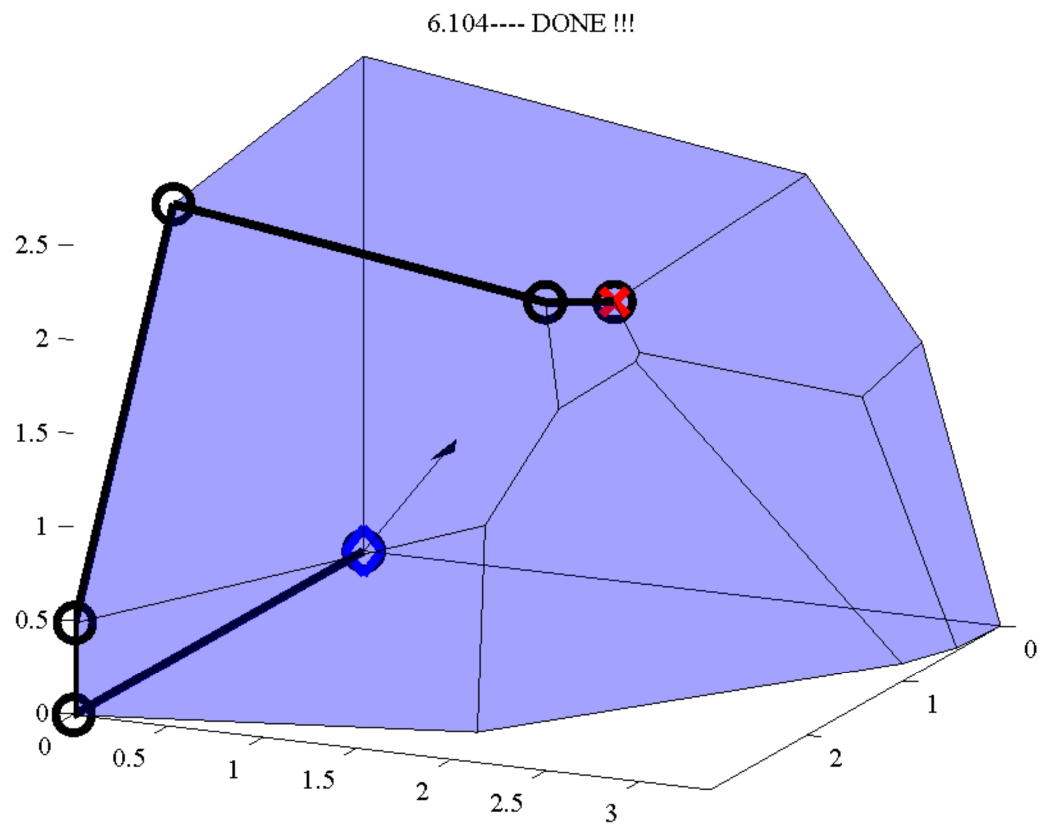
  1. Obtains an **initial** Basic feasible solution. **more on that later**.
  2. iterates: move from one BFS $\mathbf{I}$ to a **better** BFS $\mathbf{I}'$:
     - check reduced cost coefficients $c_j - c_{\mathbf{I}}^T B_{\mathbf{I}}^{-1}\mathbf{a}_j$, $j \in \mathbf{O}$. **if all negative $\mathbf{I}$ is *optimal*, OVER**.
     - otherwise, pick **one** index $\boldsymbol{e}$ for which it is positive. this will enter $\mathbf{I}$.
     - Check coordinates of $\mathbf{y}_{\boldsymbol{e}} = B_{\mathbf{I}}^{-1}\mathbf{a}_e$. **if all ≤ 0 then optimum is *unbounded*, OVER**.
     - otherwise, take the index $\boldsymbol{r}$ such that it achieves the minimum in $\{\frac{x_{i_j}}{y_{j,\boldsymbol{e}}} | y_{j,\boldsymbol{e}} > 0, 1 \leq j \leq m\}$, this will ensure feasibility. The $\boldsymbol{r}$th index of the base $\mathbf{I}$ is $i_r \leq n$.
     - $\mathbf{I}' = \{\mathbf{I} \setminus \boldsymbol{i_r}\} \cup \boldsymbol{e}$.
     - We have improved on the objective. If $x_{\mathbf{I}}$ was **not** degenerate, we have **strictly** improved.
     - $\mathbf{I} \leftarrow \mathbf{I}'$

- The loop is on a finite set of extreme points. it either exits early (unbounded), exits giving an answer (optimum $\mathbf{I}^\star$ and corresponding solution $\mathbf{x}^\star$) or loops indefinitely (degeneracy).

A Matlab Demo With Polyhedrons Containing the Origin



A Matlab Demo With Polyhedrons Containing the Origin



now with the real matlab demo...

Tableaux with Canonical Feasible Form

WHY tableaux ?

- Last time: an example where we move from a base $\mathbf{I}$ to a new base $\mathbf{I}'$, compute $B_{\mathbf{I}'}^{-1}$, do the multiplications etc.. and reach the optimum. This is the simplex.
- **Double issue**:
  - **Computational 1**: inverting matrices costs time & money. One column is different between $B_{\mathbf{I}}$ and $B_{\mathbf{I}'}$, can we do better than inverting everything again?
  - **Computational 2**: multiplying matrices costs time & money. $B_{\mathbf{I}}^{-1}A$ and $B_{\mathbf{I}'}^{-1}A$ are related.

WHY tableaux ?

- **Down to what we really need at each iteration**:
  - reduced cost coefficients vector $(c_i - z_i)$ of $\mathbf{R}^n$ to pick an index e and check optimality,
  - All column vectors of A in the base $\mathbf{I}$, that is Y, to check boundedness and choose r, namely all coordinates of $\mathbf{y}_e = B_{\mathbf{I}}^{-1} a_e$ in particular.
  - The current basic solution vector, $B_{\mathbf{I}}^{-1}\mathbf{b}$ both to choose r and on exit.
  - Having also the objective $c_{\mathbf{I}}^T B_{\mathbf{I}}^{-1}\mathbf{b}$ would help.
- Summing up, we need something that keeps track of

$\ddots \quad \cdots \quad \ddots$ $\vdots \quad B_{\mathbf{I}}^{-1}A \quad \vdots$ $\ddots \quad \cdots \quad \ddots$	$\vdots$ $B_{\mathbf{I}}^{-1}\mathbf{b}$ $\vdots$
$\cdots \quad (\mathbf{c}-\mathbf{z})' \quad \cdots$	$c_{\mathbf{I}}^T B_{\mathbf{I}}^{-1}\mathbf{b}$

Canonical Feasible Form: We know an initial BFS to corresponding Standard Form

- let's **standardize** a **feasible** $(i.e. \mathbf{b} \geq 0)$ canonical form:

$$\begin{array}{ll} \text{maximize} & \alpha^T \mathbf{u} \\ \text{subject to} & \left\{ \begin{array}{ccc} M\mathbf{u} & \leq & \mathbf{b} \\ \mathbf{u} & \geq & 0 \end{array} \right. \end{array}$$

- We assume that $\mathbf{u}, \alpha \in \mathbf{R}^d$ for a d dimensional objective and $M \in \mathbf{R}^{m \times d}$ and $\mathbf{b} \in \mathbf{R}^m$ for m constraints.

Canonical Feasible Form: We know an initial BFS to corresponding Standard Form

- Slack variables $x_{d+1}, \cdots, x_{d+m}$ can be added so that $[A, I_m] \begin{bmatrix} \mathbf{u} \\ x_{d+1} \\ \vdots \\ x_{d+m} \end{bmatrix} = \mathbf{b}$ and the problem is now with $\mathbf{c} = [\alpha, \underbrace{0, \cdots, 0}_{m}] \in \mathbf{R}^{d+m}$

$$\begin{array}{ll} \text{maximize} & x_0 = \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \left\{ \begin{array}{rcl} [M, I_m]\mathbf{x} & = & \mathbf{b} \\ \mathbf{x} & \geq & 0 \end{array} \right. \end{array}$$

- $\mathbf{x}, \mathbf{c} \in \mathbf{R}^{m+d}$, $\mathbf{c} = \left[\begin{smallmatrix} \alpha \\ \mathbf{0} \end{smallmatrix} \right]$, $A = [M, I_m] \in \mathbf{R}^{m \times (m+d)}$ and same $\mathbf{b} \in \mathbf{R}^m$.
- The dimensionality of the problem is now $n = d + m$.

Simplex Method: Tableau

Let us represent this by an (annotated) tableau:

	O						**I**						
	x_1	x_2	$\cdots$	x_e	$\cdots$	x_d	x_{d+1}	x_{d+2}	$\cdots$	x_{d+r}	$\cdots$	x_{d+m}	$\mathbf{b}$
x_{d+1}	m_{11}	m_{12}	$\cdots$	m_{1e}	$\cdots$	m_{1d}	**1**	0	$\cdots$	0	$\cdots$	0	b_1
x_{d+2}	m_{21}	m_{22}	$\cdots$	m_{2e}	$\cdots$	m_{2d}	0	**1**	$\cdots$	0	$\cdots$	0	b_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x_{d+r}	m_{r1}	m_{r2}	$\cdots$	m_{re}	$\cdots$	m_{rd}	0	0	$\cdots$	**1**	$\cdots$	0	b_r
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x_{d+m}	m_{m1}	m_{m2}	$\cdots$	m_{me}	$\cdots$	m_{md}	0	0	$\cdots$	0	$\cdots$	**1**	b_m
x_0	c_1	c_2	$\cdots$	c_e	$\cdots$	c_d	0	0	$\cdots$	0	$\cdots$	0	0

- Since $\mathbf{b} \geq 0$, take an original BFS as $\left[\underbrace{0, \cdots, 0}_{d}, b_1, b_2, \cdots, b_m\right]^T$
- Why:
  - **basic**: $I = \{d+1, \ldots, d+m\}$
  - **feasible**: $[0, \cdots, 0, b_1, b_2, \cdots, b_m]^T \geq 0$.

Simplex Method: Tableau

- the structure of the tableau so far,

A	$\mathbf{b}$
$\mathbf{c}^T$	0

- The index set $\mathbf{I}$ so far $\{d+1, d+2, \cdots, d+m\}$.
- $B_{\mathbf{I}} = I_m$, $B_{\mathbf{I}}^{-1}\mathbf{b} = \mathbf{b}$, $B_{\mathbf{I}}^{-1}A = A$ *etc.*.
- The lower-right coincides with the objective so far... 0
- $\mathbf{c}$ is actually equal to $(\mathbf{c} - \mathbf{z}_{\mathbf{I}})$ when $\mathbf{I}$ only describes slack variables.

Simplex Method without non-negativity and objectives...

- Remember: a basis $\mathbf{I}$ gives a sparse solution $\mathbf{x}_\mathbf{I}$.
- **there's one basis** $\mathbf{I}^\star$ which is the good one.
- The solution is $\mathbf{x}$ such that $\mathbf{x}_\mathbf{I}^\star = B_{\mathbf{I}^\star}^{-1}\mathbf{b}$ and the rest is zero.
- We can start with the **slack variables** as a basis in canonical **feasible** form.
- Under this form, the first matrix basis is $B_\mathbf{I} = I_m$ the identity matrix.
- We will **move** from one basis to the other. We've proved this is possible.
- In doing so, we also have to recast the cost.
- Let's check how it looks in practice, without looking at feasibility and objective related concepts.

...the Gauss pivot...

- Consider now taking a variable out of $\mathbf{I}$ to replace it by a variable in $\mathbf{O}$.
- The rth index of $\mathbf{I}$, i_r leaves the basis, e initially in $\mathbf{O}$ is removed.

...the Gauss pivot...

- Two ways of looking at the same operation:

  - Through elementary row/column operations transfer a vector $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ where the 1 is in position r to get a similar basis vector in the eth column of A.

$$\begin{array}{c} \\ \vdots \\ r \\ \vdots \\ i \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \cdots & i_r & \cdots & e & \cdots \end{array} \\ \begin{bmatrix} \ddots & 0 & \ddots & \vdots & \ddots \\ \cdots & 1 & \cdots & a_{re} & \cdots \\ \ddots & 0 & \ddots & \vdots & \ddots \\ \cdots & 0 & \cdots & a_{ie} & \cdots \\ \ddots & 0 & \ddots & \vdots & \ddots \end{bmatrix} \end{array} \Rightarrow \begin{array}{c} \\ \vdots \\ r \\ \vdots \\ i \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} \cdots & i_r & \cdots & e & \cdots \end{array} \\ \begin{bmatrix} \ddots & \vdots & \ddots & 0 & \ddots \\ \cdots & a_{r,i_r} & \cdots & \mathbf{1} & \cdots \\ \ddots & \vdots & \ddots & 0 & \ddots \\ \cdots & a_{i,i_r} & \cdots & 0 & \cdots \\ \ddots & \vdots & \ddots & 0 & \ddots \end{bmatrix} \end{array}$$

...the Gauss pivot...

- Consider the equalities $A\mathbf{x} = \mathbf{b}$ written in row form,

$$\mathbf{u}_i^T\mathbf{x} = b_i$$

where the $\mathbf{u}$'s are the rows of A.

- Putting variable x_e in the basis is equivalent to isolating x_e so that is present in **all but one** of the m equations, with coefficient 1. On the other hand we let x_{i_r} enter all equations again, that is

$$x_e = \widetilde{b_i} - \sum_{i=1, i\neq e}^{n} \rho_i x_i$$

and x_e does not appear elsewhere.

...the Gauss pivot

- This is achieved through a pivot in the tableau.
- Once the rth element of basis $\mathbf{I}$, namely column $i_r \leq n$, and $e \leq n$ are agreed upon, the rules to update the tableau are:

(a) in pivot row $a_{rj} \leftarrow a_{rj}/a_{re}$.
(b) in pivot column $a_{re} \leftarrow 1, a_{ie} = 0$ for $i = 1, \cdots, m, i \neq r$: the eth column becomes a matrix of zeros and a one.
(c) for all other elements $a_{ij} \leftarrow a_{ij} - \frac{a_{rj}a_{ie}}{a_{re}}$

The Gauss pivot

- Graphically,

$$\begin{array}{c} \\ \vdots \\ i \\ \vdots \\ r \\ \vdots \end{array}\begin{array}{c} \begin{array}{ccccc} \cdots & j & \cdots & e & \cdots \end{array} \\ \begin{bmatrix} \ddots & \vdots & \ddots & \vdots & \ddots \\ \cdots & a_{ij} & \cdots & \boldsymbol{a_{ie}} & \cdots \\ \ddots & \vdots & \ddots & \vdots & \ddots \\ \cdots & \boldsymbol{a_{rj}} & \cdots & \boldsymbol{a_{re}} & \cdots \\ \ddots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}\end{array} \Rightarrow \begin{array}{c} \\ \vdots \\ i \\ \vdots \\ r \\ \vdots \end{array}\begin{array}{c} \begin{array}{ccccc} \cdots & j & \cdots & e & \cdots \end{array} \\ \begin{bmatrix} \ddots & \vdots & \ddots & \vdots & \ddots \\ \cdots & a_{ij} - \frac{\boldsymbol{a_{rj}a_{ie}}}{\boldsymbol{a_{re}}} & \cdots & 0 & \cdots \\ \ddots & \vdots & \ddots & \vdots & \ddots \\ \cdots & \boldsymbol{a_{rj}}/\boldsymbol{a_{re}} & \cdots & \mathbf{1} & \cdots \\ \ddots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}\end{array}$$

- Look at how the column e is now a column of 0 and 1's. This makes sense since $B_{\mathbf{I}}^{-1}\mathbf{a}_e = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ with 1 in eth position means $\mathbf{a}_e$ is in the basis.

Linear system and pivoting

- Consider the linear system

$$\left\{\begin{array}{rcrcrcrcrcrcl} x_1 & + & x_2 & - & x_3 & + & x_4 & & & & & = & 5 \\ 2x_1 & - & 3x_2 & + & x_3 & & & + & x_5 & & & = & 3 \\ -x_1 & + & 2x_2 & - & x_3 & & & & & + & x_6 & = & 1 \end{array}\right.$$

- The corresponding tableau

$$\begin{array}{ccccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{b} \end{array}$$
$$\left[\begin{array}{ccccccc} 1 & 1 & -1 & 1 & 0 & 0 & 5 \\ 2 & -3 & 1 & 0 & 1 & 0 & 3 \\ -1 & 2 & -1 & 0 & 0 & 1 & 1 \end{array}\right]$$

Simplex Method: Swapping Indexes

- in the corresponding tableau,

$$\begin{array}{c} \\ \mathbf{a}_4 \\ \mathbf{a}_5 \\ \mathbf{a}_6 \end{array}\begin{array}{ccccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{b} \end{array}$$

$$\begin{bmatrix} 1 & 1 & -1 & \mathbf{1} & 0 & 0 & 5 \\ 2 & -3 & 1 & 0 & \mathbf{1} & 0 & 3 \\ -1 & 2 & -1 & 0 & 0 & \mathbf{1} & 1 \end{bmatrix}$$

notice the structure:

$$\boxed{\begin{array}{ccccccc} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & M & \vdots & \vdots & I_3 & \vdots & \mathbf{b} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \end{array}}$$

- And the fact that by taking the obvious basis $\mathbf{I} = \{4, 5, 6\}$ we have $B_{\mathbf{I}} = I_3$ and $B_{\mathbf{I}}^{-1} = I_3$

Simplex Method: Let's pivot

- Let's pivot arbitrarily. We put **1** in the base and remove **4**.

$$\begin{array}{c} \\ \mathbf{a}_4 \\ \mathbf{a}_5 \\ \mathbf{a}_6 \end{array}\begin{array}{c} \begin{array}{ccccccc} \boldsymbol{x}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{b} \end{array} \\ \left[\begin{array}{ccccccc} 1 & 1 & -1 & 1 & 0 & 0 & 5 \\ 2 & -3 & 1 & 0 & 1 & 0 & 3 \\ -1 & 2 & -1 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

which yields

$$\begin{array}{c} \\ \mathbf{a}_1 \\ \mathbf{a}_5 \\ \mathbf{a}_6 \end{array}\begin{array}{c} \begin{array}{ccccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{b} \end{array} \\ \left[\begin{array}{ccccccc} \mathbf{1} & 1 & -1 & 1 & 0 & 0 & 5 \\ \mathbf{0} & -5 & 3 & -2 & 1 & 0 & -7 \\ \mathbf{0} & 3 & -2 & 1 & 0 & 1 & 6 \end{array}\right] \end{array}$$

- $\mathbf{I} = \{1, 5, 6\}$, that is $B_{\mathbf{I}} = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{smallmatrix}\right]$. The basic solution is such that $\mathbf{x}_{\mathbf{I}} = B_{\mathbf{I}}^{-1}\mathbf{b}$
- Note that all coordinates of $\mathbf{a}_1, \cdots, \mathbf{a}_6, \mathbf{b}$ in the table are given with respect to $\mathbf{a}_1, \mathbf{a}_5, \mathbf{a}_6$. In particular the last column corresponds to $B_{\mathbf{I}}^{-1}\mathbf{b}$...not feasible here BTW.

Simplex Method: again...

- Let's pivot arbitrarily again, this time inserting **2** and removing the **second** variable of the basis, **5**.

$$\begin{array}{c} \\ \mathbf{a}_1 \\ \mathbf{a}_5 \\ \mathbf{a}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{b} \end{array} \\ \left[\begin{array}{ccccccc} 1 & 1 & -1 & 1 & 0 & 0 & 5 \\ 0 & \mathbf{-5} & 3 & -2 & 1 & 0 & -7 \\ 0 & 3 & -2 & 1 & 0 & 1 & 6 \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{b} \end{array} \\ \left[\begin{array}{ccccccc} 1 & \mathbf{0} & -\frac{2}{5} & \frac{3}{5} & \frac{1}{5} & 0 & \frac{18}{5} \\ 0 & \mathbf{1} & -\frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 & \frac{7}{5} \\ 0 & \mathbf{0} & -\frac{1}{5} & -\frac{1}{5} & \frac{3}{5} & 1 & \frac{9}{5} \end{array}\right] \end{array}$$

- Notice how one can keep track of who is in the basis by checking where 0/1's columns are.

- The solution is now feasible... pure luck.

Simplex Method: and again...

- once again, pivot inserting **3** and removing the **third** variable of the basis, **6**.

$$\begin{array}{c} \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_6 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{b} \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & -\frac{2}{5} & \frac{3}{5} & \frac{1}{5} & 0 & \frac{18}{5} \\ 0 & 1 & -\frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 & \frac{7}{5} \\ 0 & 0 & -\frac{1}{5} & -\frac{1}{5} & \frac{3}{5} & 1 & \frac{9}{5} \end{array}\right] \end{array}$$

$$\begin{array}{c} \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \mathbf{b} \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & \mathbf{0} & 1 & -1 & -2 & 0 \\ 0 & 1 & \mathbf{0} & 1 & -2 & -3 & -4 \\ 0 & 0 & \mathbf{1} & 1 & -3 & -5 & -9 \end{array}\right] \end{array}$$

- horrible. moving randomly we have a now non-feasible degenerate basic solution.
- yet we knew that pivoting randomly based only on $y_{r,e} \neq 0$ would lead us nowhere.

Adding the reduced costs

- What happens when we also pivot the last line?
- Remember the last line is equal to $\mathbf{v} \stackrel{\text{def}}{=} (\mathbf{c} - \mathbf{z})'$ in the beginning.
- Remember also that

(a) in pivot row $a_{rj} \leftarrow a_{rj}/a_{re}$.
(b) in pivot column $a_{re} \leftarrow 1, a_{ie} = 0$ for $i = 1, \cdots, m, i \neq r$: the eth column becomes a matrix of zeros and a one.
(c) for all other elements $a_{ij} \leftarrow a_{ij} - \frac{a_{rj}a_{ie}}{a_{re}}$

- Here, (a) does not apply, we cannot be in the pivot row.
- we have
  - in pivot column $v_e = 0$: makes sense, reduced cost is zero for basis elements.
  - for all other elements $v_j \leftarrow v_j - \frac{a_{rj}v_e}{a_{re}}$.

- Recapitulating, at each iteration of the pivot the matrix is exactly

$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\vdots$
$\vdots$	$B_{\mathbf{I}}^{-1}M$	$\vdots$	$\vdots$	$B_{\mathbf{I}}^{-1}$	$\vdots$	$B_{\mathbf{I}}^{-1}\mathbf{b}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\vdots$
$\dots$		$(\mathbf{c}-\mathbf{z})$			$\dots$	$-x_0$

- The pivot is thus applied on the $m+1 \times n+1$ tableau.
- The tableau contains **everything we need**, reduced costs, (minus)objective, the coordinates of $B_{\mathbf{I}}^{-1}\mathbf{b}$ and $B_{\mathbf{I}}^{-1}A$

Tableaux with Arbitrary Initial BFS

Working around to go back to previous situation

- Suppose that we are given an arbitrary BFS $\mathbf{I}$ for the problem

$$\begin{array}{ll} \text{maximize} & x_0 = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} A\mathbf{x} & = & \mathbf{b} \\ \mathbf{x} & \geq & 0 \end{cases} \end{array}$$

- We try to go back to the previous situation.

Working around to go back to previous situation

- Perform a permutation of columns such that columns in positions $i_1, \cdots, i_m$ become in last positions $n-m+1, \cdots, n$.
- A is now $[N, B]$ (N for non-basic part) and the system can be written as

$$\begin{cases} N\mathbf{x}_N & + & B\mathbf{x}_B & = \mathbf{b} \\ \mathbf{c}_N^T\mathbf{x}_N & + & \mathbf{c}_B^T\mathbf{x}_B & = x_0 \end{cases}$$

- Multiplying the first line by B^{-1},

$$B^{-1}N\mathbf{x}_N + \mathbf{x}_B = B^{-1}\mathbf{b} \text{ thus } \mathbf{x}_B = B^{-1}\mathbf{b} - B^{-1}N\mathbf{x}_N$$

which when used for objective x_0 yields

$$x_0 = (\mathbf{c}_N - \mathbf{c}_B^T B{-}1N)^T\mathbf{x}_N + c_b^T B^{-1}\mathbf{b}$$

Working around to go back to previous situation

- We can now use the same tableau:

$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\vdots$
$\vdots$	$B^{-1}N$	$\vdots$	$\vdots$	I	$\vdots$	$\mathbf{b}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\vdots$
$\dots$	$\mathbf{c}_N^T - \mathbf{z}_N^T$	$\dots$	$\dots$	$\mathbf{0}$	$\dots$	$-x_0$

- And can apply the simplex as defined for canonical feasible forms.

Short Comment on Dictionaries

- A dictionary is a comparable compact form

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & \zeta & = & & & & & 4x_2 & + & 3x_3 \\
\text{subject to} & w_1 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
 & w_2 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
 & w_3 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
 & & & \multicolumn{5}{r}{x_1,\ x_2,\ x_3,\ w_1,\ w_2,\ w_3} & \geq & 0.
\end{array}
$$

  where basic variables are kept on the top and non-basic are kept on the left.

- We save space (1/0 columns) but need to **keep track of variable names**.
- The constants on the left correspond to the last column in tableaux.
- The first line stands for reduced cost coefficients of **nonbasic variables**.
- The lower-right corresponds to minus the $B_{\mathbf{I}}^{-1}A$ matrix for indices in $\mathbf{O}$.
- Equivalent to Tableaux, rather used for educational purposes.