ICML 6/17

Mean Reversion with a Variance Threshold

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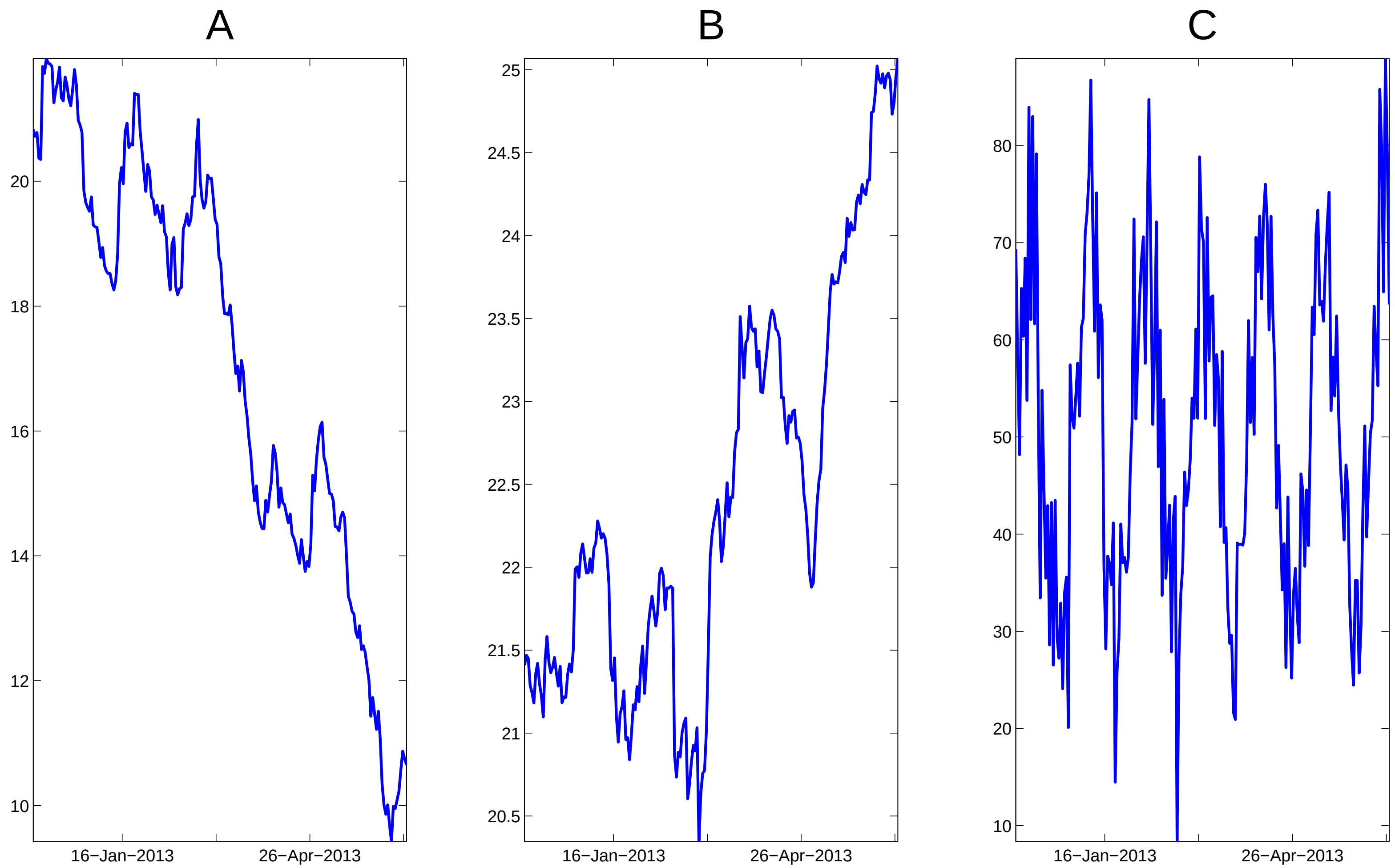
Motivation

Assets



A
B
C
20
18
16
14
12
10
25
24.5
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22.5
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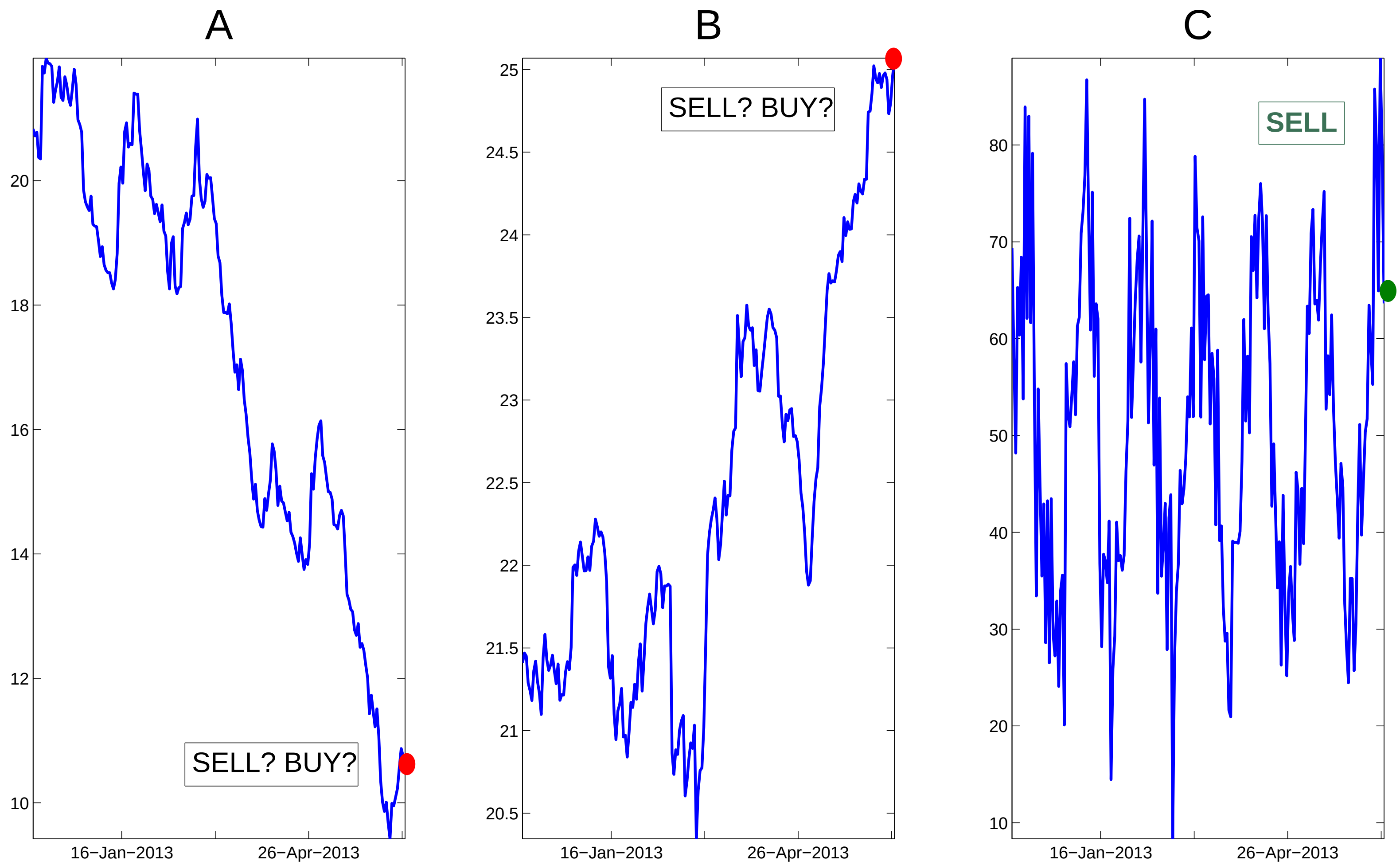
20.5
80
70
60
50
40
30
20
10
16−Jan−2013
26−Apr−2013

Assets



A
B
C
SELL? BUY?
SELL? BUY?
SELL
20
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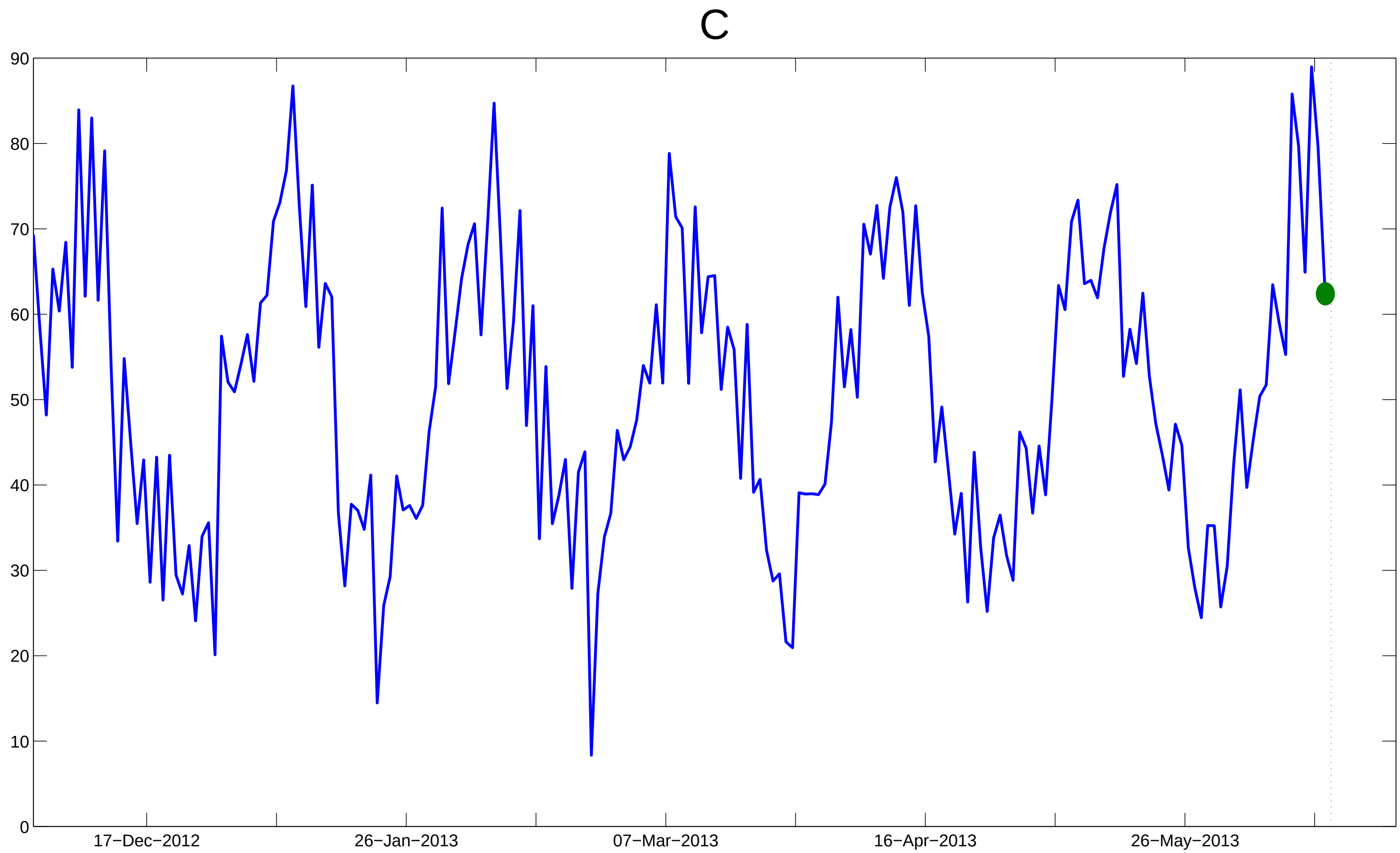
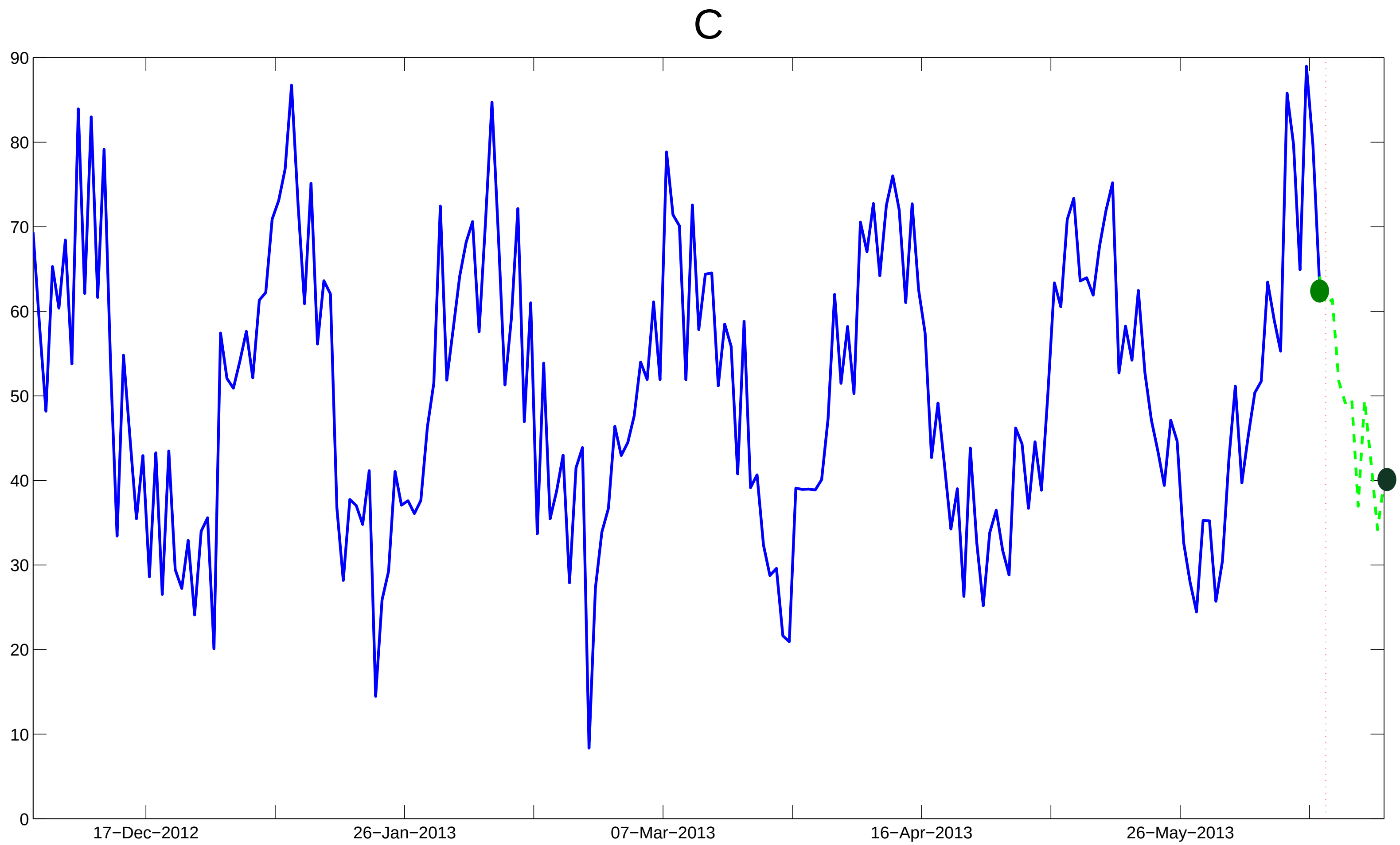
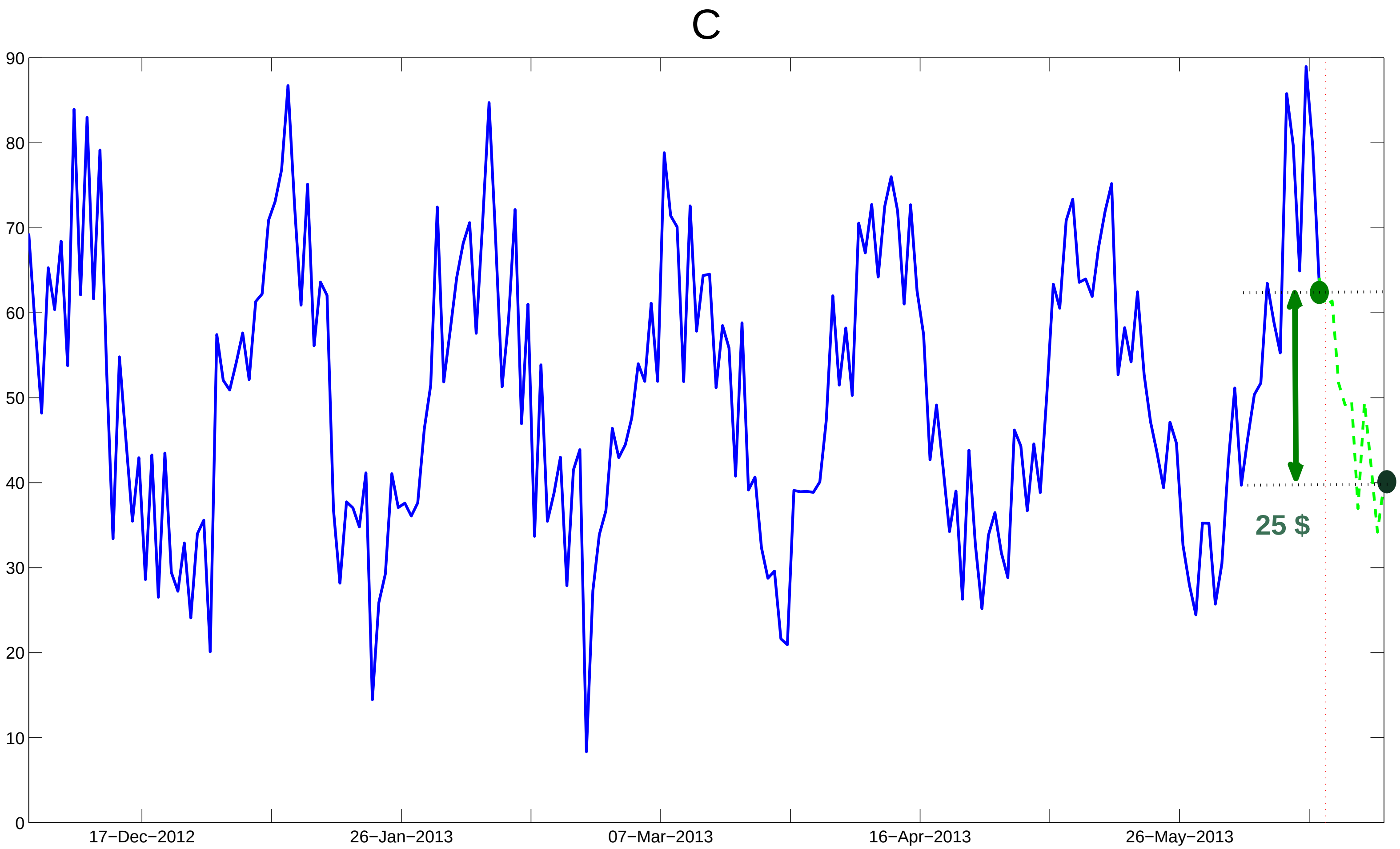
Mean Reverting Trade



Mean Reverting Trade



Mean Reverting Trade



Mean Reverting Process

$$x_t \in \mathbb{R}_+$$

price of **single** asset at time t

- Mean-reversion $=$ tendency to pull back to mean
- First-order Stationary processes are mean-reverting

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Most financial assets are not mean reverting on a short time horizon

in fact, most financial assets are **not** stationary

Purpose of Cointegration

$$\mathbf{x}_t = \begin{bmatrix} x_{1,t} \\ \vdots \\ x_{d,t} \end{bmatrix} \in \mathbb{R}^d_+, d \text{ assets at time } t$$

- Cointegration: find $y \in \mathbb{R}^d$ s.t. $y^T \mathbf{x}_t$ is **stationary**.

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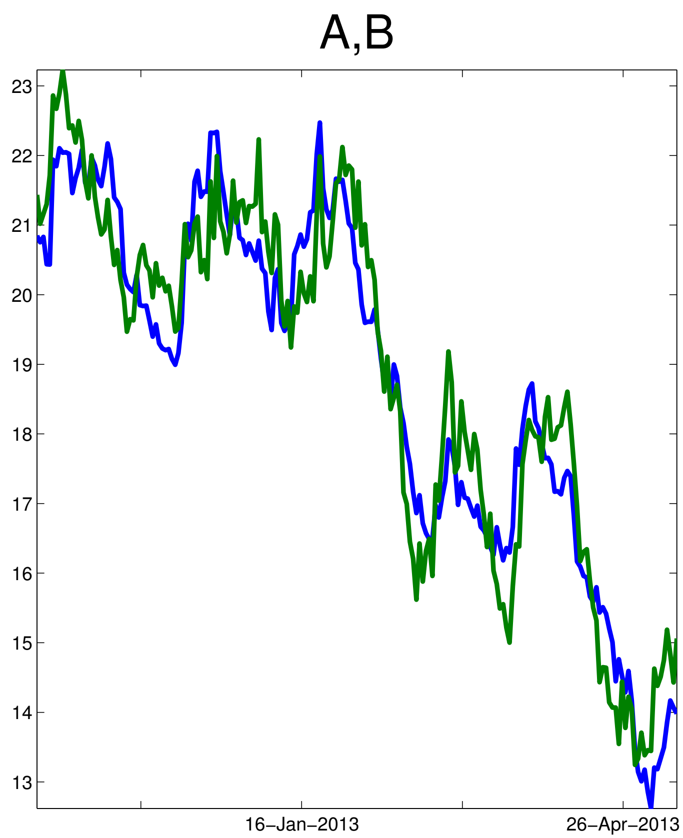
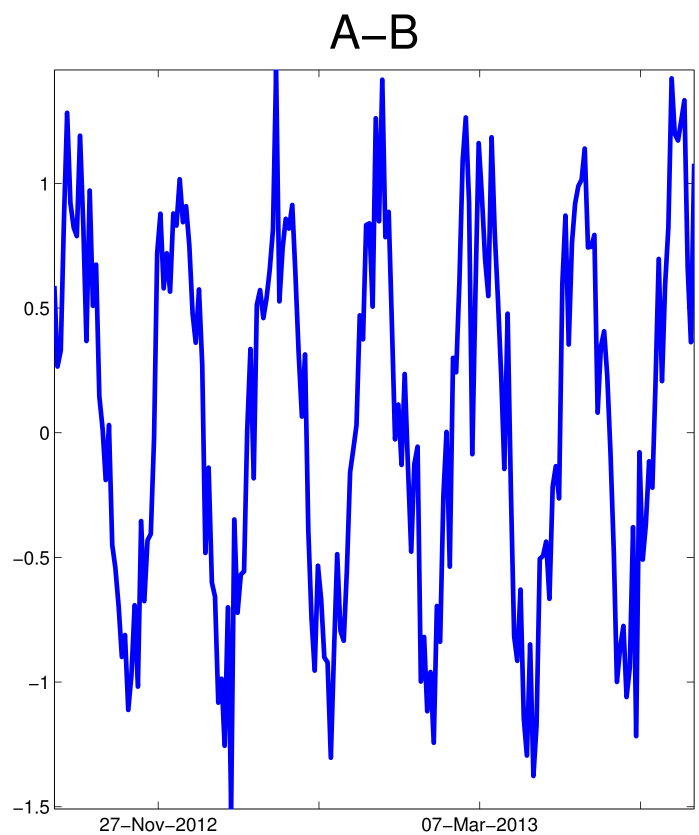
- Cointegration: find $y \in \mathbb{R}^d$ s.t. $y^T\mathbf{x}_t$ is **stationary**.
- Straightforward application to finance:

1. Estimate y using historical data
2. trade basket $(y^T\mathbf{x}_t)$ as a mean reverting asset.

Purpose of Cointegration





$d = 2$: Pairs trading. $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Estimate Cointegrated Relationships

- Existing work
  - Define a criterion to measure non-stationarity

$$\lambda(y) = \text{non-stationarity}(y^T \mathbf{x}_t)$$

  - Minimize $\lambda(y)$.

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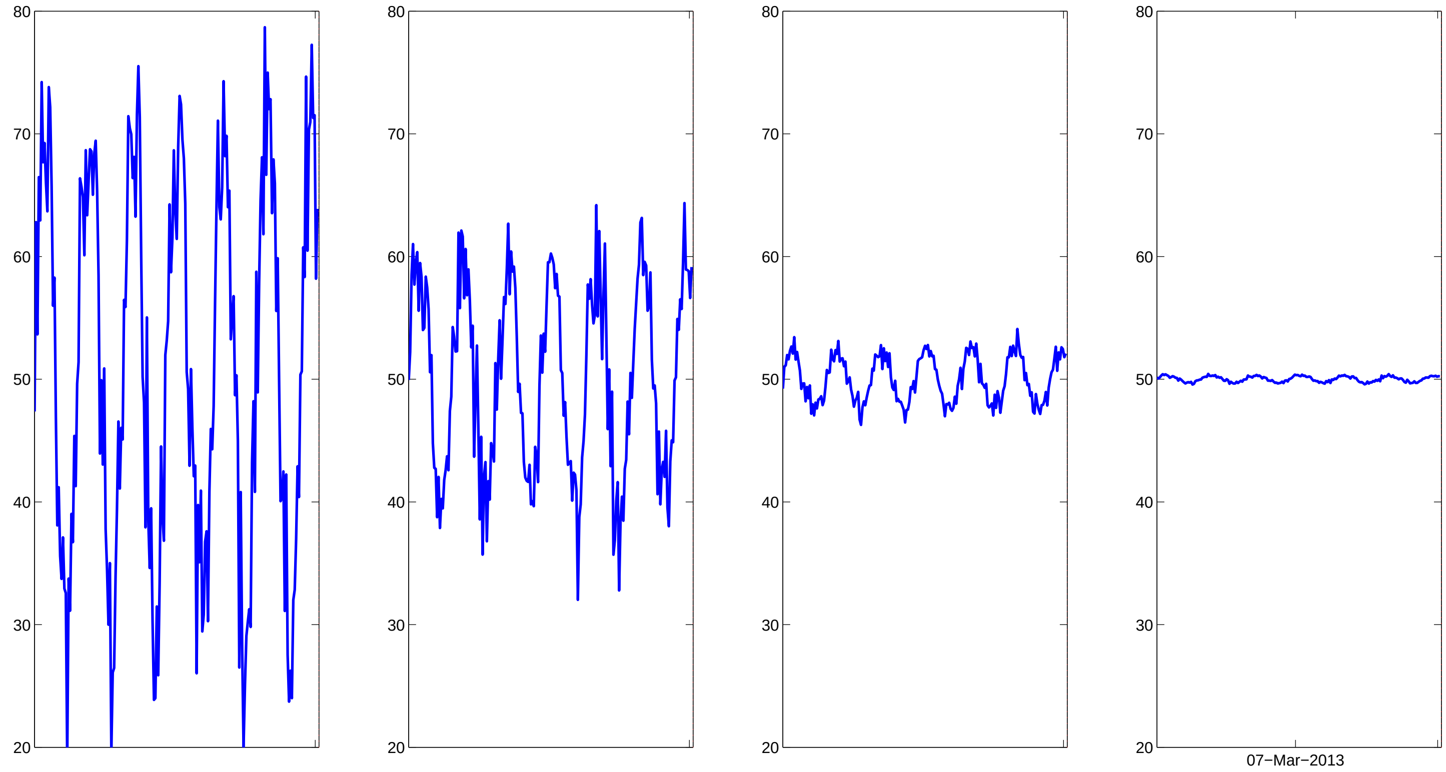
  - Minimize $\lambda(y)$.
- λ typically defined using time series modeling (VEC)
- Nobel prize in economics Engle / Granger in 2003.
- Well studied topic in econometrics

Contribution

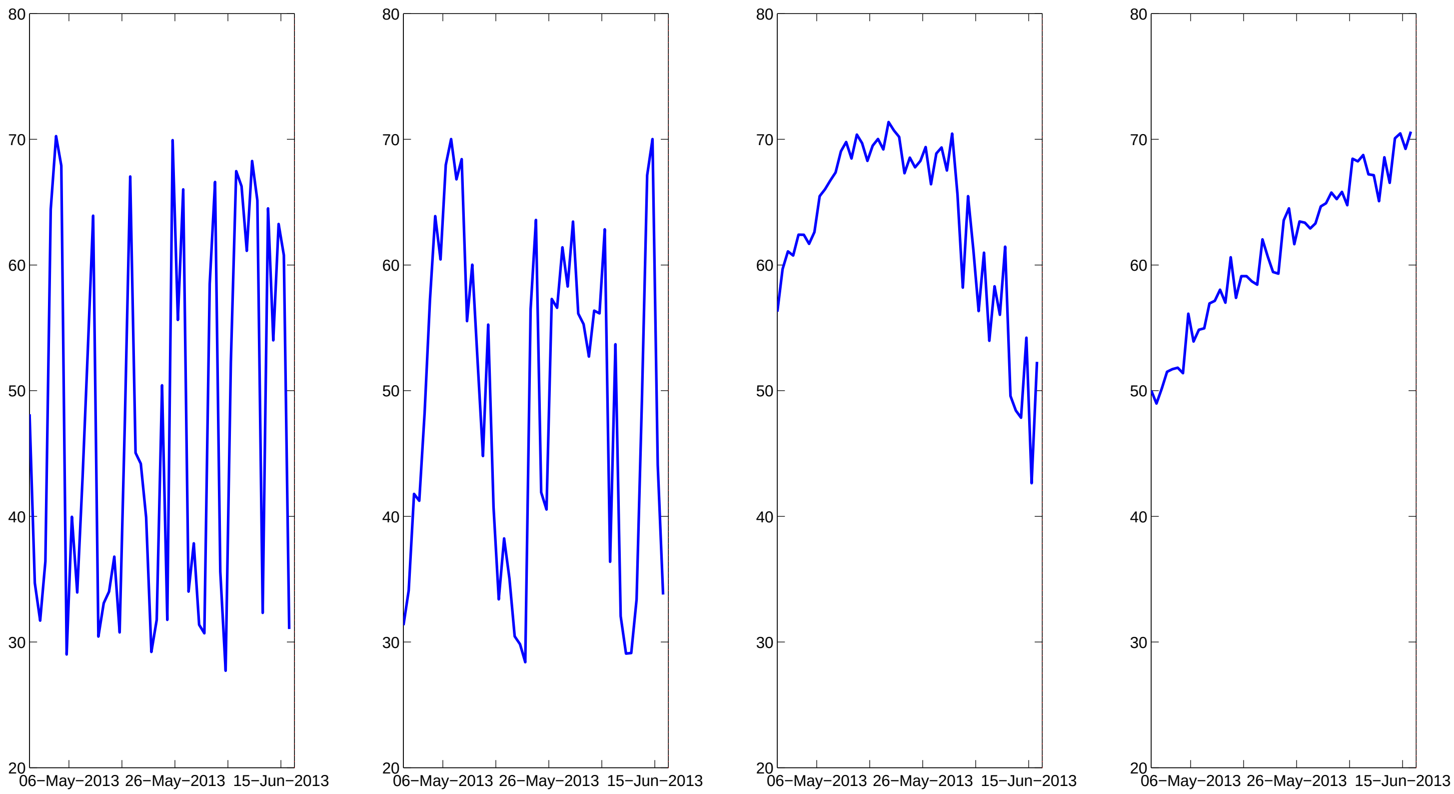
Stationarity is not enough

All Mean Reverting



The more variance $y^T \mathbf{x}_t$ has, the better

All Mean Reverting



The faster $y^T \mathbf{x}_t$ mean reverts, the better

Observation

Mean reverting assets that have

- **small variance**
- **slow mean reversion**

require **more leverage** to reach the same level of profit.

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Problem? cointegration techniques consider neither

Our Contribution

- Similar starting point:
  - Define criteria to measure mean-reversion

$$\lambda(y) = \text{slow-mean-reversion}(y^T \mathbf{x}_t)$$

- Take into account the **variance** of $y^T \mathbf{x}_t$
  - Minimize $\lambda(y)$ subject to $\mathbf{var}(y^T \mathbf{x}_t) \geq \nu$

Criteria

We consider 3 new criteria

- **Portmanteau statistic**: minimize autocorrelations
- **Expected crossings**: maximize the expected number of mean crossings using crossing statistics
- **Predictability**: reuse older work by Box/Tiao '77

Use semidefinite programming to optimize them

Portmanteau Criterion

- Portmanteau statistic of univariate process

$$\text{por}_p(x) = \frac{1}{p}\sum_{i=1}^{p}\left(\frac{\mathbf{E}[x_t x_{t+i}]}{\mathbf{E}[x_t^2]}\right)^2$$

- $\propto$ Euclidean norm of autocorrellogram coefficients

- Used to test if a process is *white noise* (Ljung-Box).

Portmanteau Criterion

- If $\mathbf{x}_t \in \mathbb{R}^d$, consider for $y \in \mathbb{R}^d$

$$\mathrm{por}_p(y^T \mathbf{x}_t) = \frac{1}{p} \sum_{i=1}^{p} \left(\frac{y^T A_i y}{y^T A_0 y} \right)^2,$$

where

$$A_i \stackrel{\mathrm{def}}{=} \frac{1}{T-i-1} \sum_{t=1}^{T-i} \tilde{\mathbf{x}}_t \tilde{\mathbf{x}}_{t+i}^T$$

Portmanteau Criterion

- Minimizing $\text{por}_p(y^T \mathbf{x}_t)$ under variance constraint:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{p} \left(y^T A_i y\right)^2 \\ \text{subject to} & \boldsymbol{y^T A_0 y \geq \nu} \\ & \|y\|_2 = 1. \end{array} \qquad \text{(P)}$$

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- This problem is not convex

Semidefinite Relaxations

- Change of variables: $Y = yy^T$
- Trick: cast the problem as a **semidefinite program**

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{p} \mathbf{Tr}(A_i Y)^2 \\ \text{subject to} & \mathbf{Tr}(A_0 Y) \geq \nu \\ & \mathbf{Tr}(Y) = 1,\ Y \succeq 0 \end{array} \qquad \text{(SDP)}$$

- With the extra constraint $\mathbf{Rank}(Y) = 1$, (SDP) and (P) are **equivalent**.

$\mathcal{S}$-lemma

- Brickman'61: when $d \geq 3$ and for two matrices A, B

$$\left\{(y^T Ay, y^T By) : y \in \mathbb{R}^d,\ \|y\|_2 = 1\right\} =$$

$$\{(\mathbf{Tr}(AY), \mathbf{Tr}(BY)) : Y \in \mathbf{S}_d,\ \ \mathbf{Tr}\, Y = 1,\ Y \succeq 0\}$$

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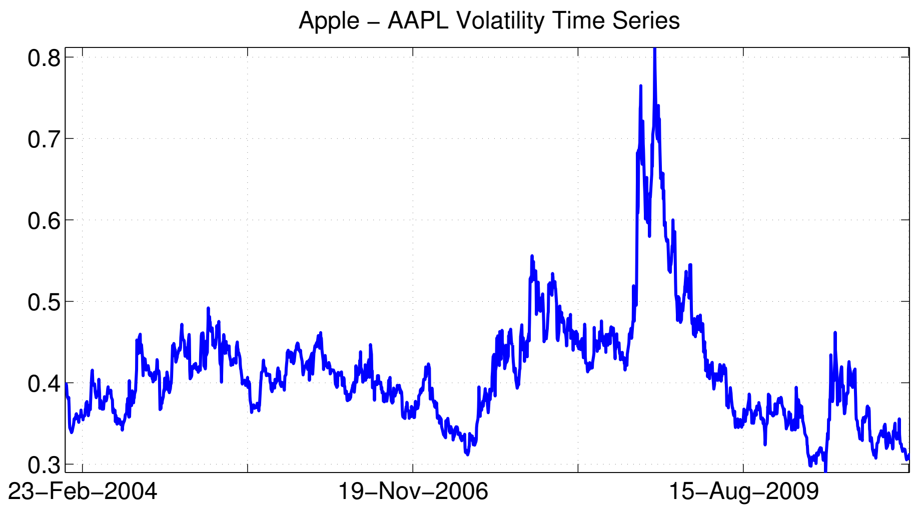
$$\{(\mathbf{Tr}(AY), \mathbf{Tr}(BY)) : Y \in \mathbf{S}_n,\ \ \mathbf{Tr}\, Y = 1,\ Y \succeq 0\}$$

- When $p = 1$, the relaxation is **exact**: $Y^\star \to y^\star$
- When $p > 2$, one can find an approximate solution $\tilde{y}$ using $Y^\star$ with suboptimality guarantees.

Experiments

Implied Volatility



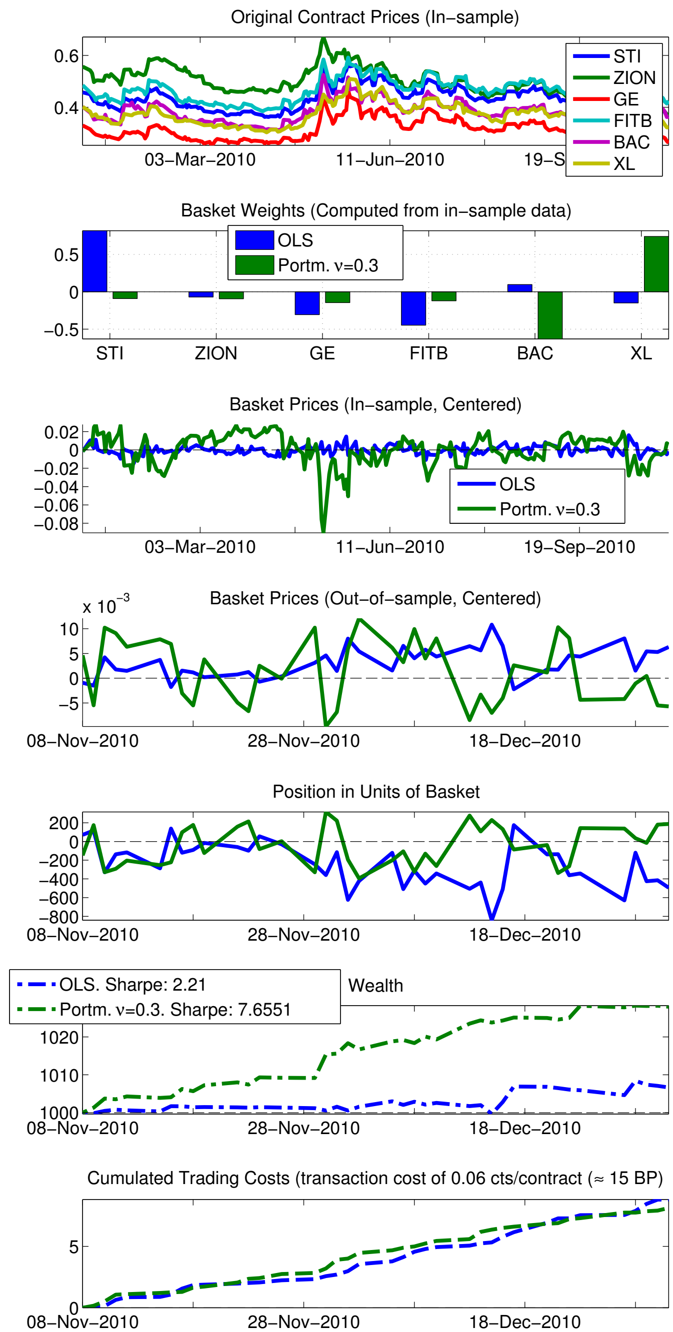
- Volatility data for 217 stocks over 8 years
- Clustered in 13 sectors
- Divided into 20 time windows, 85% train, 15% test
- Greedy selection of 50 baskets per window

Comparisons with Classic Cointegration

- Orthogonal Least Squares: smallest eigenvector of A_0
- Fully Modified OLS (Phillips'95)
- Johansen VEC model (Johansen'92)

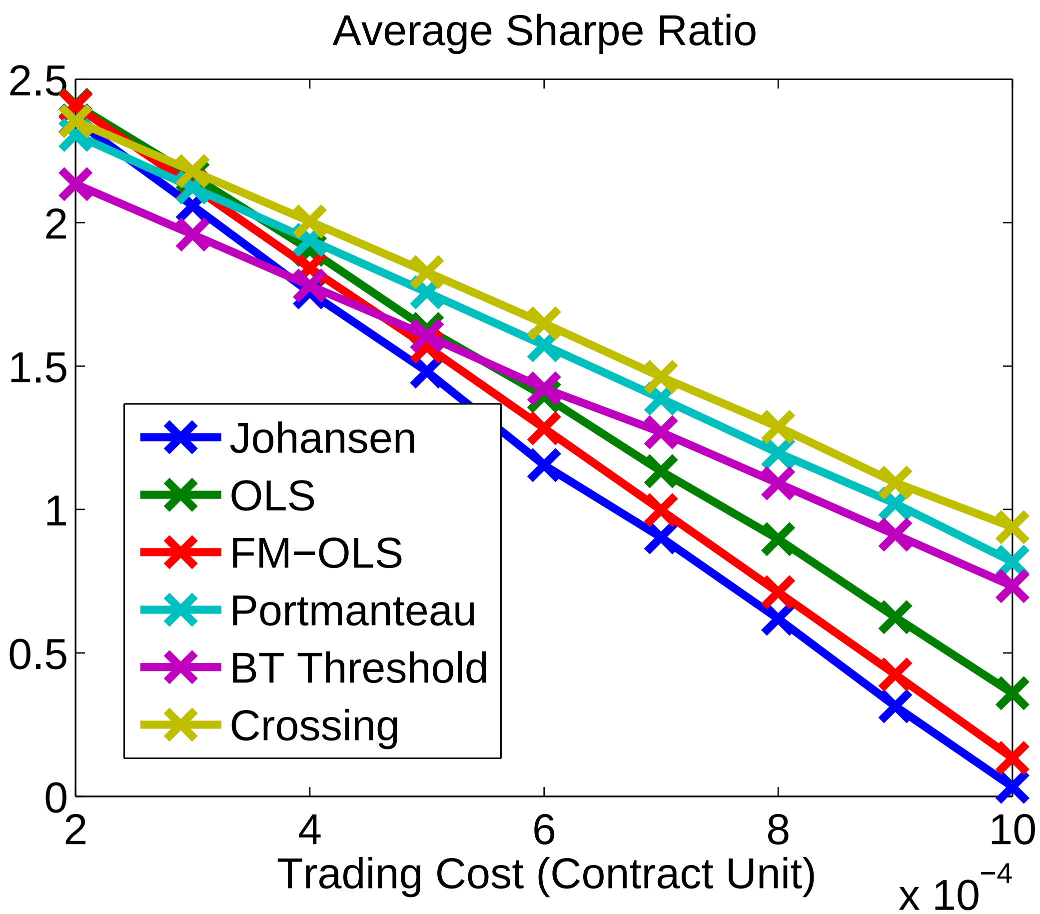
Our criteria, with $\nu = 0.3 \times \text{median}(\mathbf{var}\, x_i)$

- Portmanteau
- Crossing Stats
- Predictability



Original Contract Prices (In−sample)
0.6
0.4
03−Mar−2010
11−Jun−2010
19−S
STI
ZION
GE
FITB
BAC
XL
Basket Weights (Computed from in−sample data)
OLS
Portm. ν=0.3
0.5
0
−0.5
STI
ZION
GE
FITB
BAC
XL
Basket Prices (In−sample, Centered)
0.02
0
−0.02
−0.04
−0.06
−0.08
OLS
Portm. ν=0.3
03−Mar−2010
11−Jun−2010
19−Sep−2010
Basket Prices (Out−of−sample, Centered)
x 10−3
10
5
0
−5
08−Nov−2010
28−Nov−2010
18−Dec−2010
Position in Units of Basket
200
0
−200
−400
−600
−800
08−Nov−2010
28−Nov−2010
18−Dec−2010
OLS. Sharpe: 2.21
Portm. ν=0.3. Sharpe: 7.6551
Wealth
1020
1010
1000
08−Nov−2010
28−Nov−2010
18−Dec−2010
Cumulated Trading Costs (transaction cost of 0.06 cts/contract (≈ 15 BP)
5
0
08−Nov−2010
28−Nov−2010
18−Dec−2010

Sharpe



$8 \times 10^{-4} \approx 20$ Basis points

Influence of Variance Threshold ν